Support Hull of a Polynomial System and the Dixon-based Resultant Construction[1]

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Abstract

[KSZ92] showed that the degree of the toric resultant of a generic polynomial system consisting of $n+1$ polynomials in n variables (over the toric variety) is determined by the mixed volume of the convex hull of the associated supports. In practice, it is however not possible to construct resultant matrices exactly so that their determinants are precisely resultants. Usually, the determinant of a resultant matrix includes extraneous information. It can be shown that generic inclusion of monomials corresponding to lattice points in the convex hull of a given support can contribute to the extraneous factors in the determinant computation of the associated resultant matrix (since the degree of the resultant is not affected due to no change in the convex hull of the support). Until recently, how monomials corresponding to lattice points inside in the convex hull of a given support affect the resultant computation is not well understood.

The concept of a **support hull** induced by the support of a polynomial system is defined. The support hull can be shown to determine the degree of the determinant of resultant matrices generated using the Dixon resultant formulation. The support hull is contained within the convex hull of a given support. The concept of a **support hull interior point** is defined relative to the support hull of a given support. It is shown that generic inclusion of monomials corresponding to support hull interior points in a polynomial system does not contribute to the degree of the determinant of the corresponding (both Dixon and Dixon-dialytic) resultant matrices. However, generic inclusion of a monomial corresponding to a lattice point inside the convex hull but not in the support hull, in a given polynomial system contributes to the degree of the determinant of the associated resultant matrices, and hence, to increasing the degree of the extraneous factor.

It is conjectured that the concepts of support hull and support-hull interior points are likely to be useful for analyzing extraneous factors for resultant formulations other than the Dixon formulation as well.

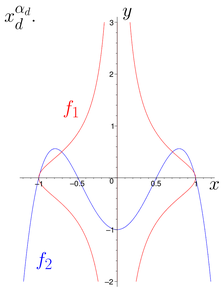
The above analysis is done for unmixed polynomial systems, i.e., all polynomials have the same support. However, the concept of support hull is useful for computing resultants of mixed systems as well. The supports of polynomials in a mixed system can be translated so as to minimize the support hull of the union of the supports of the polynomials.

Keywords: Elimination Theory, Polynomial system solving, Toric variety, Resultant computation, Dixon resultant formulation, Bézoutians, Convex Hull, Mixed Volume, Support Hull, Extraneous factors

1 Polynomial system and its Resultant

Consider a polynomial system

$$\mathcal{F} = \begin{cases} f_0 = \sum_{\alpha\in\mathcal{A}_0} c_{0,\alpha}\,\mathbf{x}^\alpha, \\ f_1 = \sum_{\alpha\in\mathcal{A}_1} c_{1,\alpha}\,\mathbf{x}^\alpha, \\ \vdots \\ f_d = \sum_{\alpha\in\mathcal{A}_d} c_{d,\alpha}\,\mathbf{x}^\alpha, \end{cases} \quad \text{where} \quad \mathbf{x}^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}.$$



For example,

$$\mathcal{F} = \begin{cases} f_0 = ax + by + c, \\ f_1 = x^3 + 4x^3y^2 - x, \\ f_2 = y + 1 + 4x^4 - 5x^2, \end{cases}$$

Its resultant is

$$\texttt{Res}(f_0, f_1, f_2) = C \prod_{\varepsilon\in\mathcal{Z}(f_1,f_2)} f_0(\varepsilon)$$

where C is independent of the coefficients of f_0.

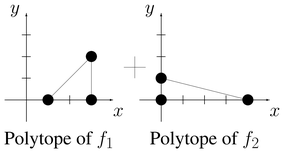
The degree of the resultant is determined by the number of solutions.

2 Number of solutions

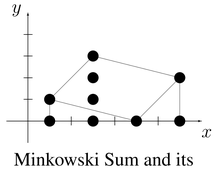
- **Projective solutions**: 20;
  this is determined by the *total degree* of polynomials (Bézout Theorem)
  $$\deg f_1 \ \deg f_2 = 5 \times 4 = 20$$
- **Affine**: 11 (determined by the convex hulls and their positions)
- **Toric**: 8;
  (determined by the mixed volume of the convex hulls/BKK bound)

$$\begin{aligned}\mu(\mathcal{A}_1, \mathcal{A}_2) &= \texttt{Vol}(\mathcal{A}_1 + \mathcal{A}_2) - \texttt{Vol}(\mathcal{A}_1) - \texttt{Vol}(\mathcal{A}_2) \\ &= 18 - 4 - 4 = 10.\end{aligned}$$

The mixed volume μ gives an upper bound on the number of toric roots. The bound is tight for generic systems.



Polytope of f_1 Polytope of f_2



Minkowski Sum and its subdivision

- Mixed volume thus "determines" the degree of the toric resultant.
- Mixed volume is invariant under the presence of convex hull interior points.
- The degree of the toric resultant is invariant under the presence of monomials corresponding to the lattice points in the convex hull interior. [KSZ92, Emi94]

3 Extraneous factors in matrix-based resultant methods

Sylvester-type constructions

$$M = \left(\begin{array}{cccccccccccccccc}
0&0&0&a&0&b&c&0&0&0&0&0&0&0&0&0\\
0&0&0&0&a&0&b&c&0&0&0&0&0&0&0&0\\
0&0&0&0&0&a&0&b&c&0&0&0&0&0&0&0\\
0&0&0&0&0&0&a&0&b&c&0&0&0&0&0&0\\
0&0&0&0&0&0&0&a&0&b&c&0&0&0&0&0\\
0&0&0&0&0&0&0&0&a&0&b&c&0&0&0&0\\
0&0&0&0&0&0&0&0&0&a&0&b&c&0&0&0\\
0&0&0&0&0&0&0&0&0&0&a&0&b&c&0&0\\
0&0&0&0&0&0&0&0&0&0&0&a&0&b&c&0\\
\hline
4&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&4&0&1&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&4&0&1&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&4&0&1&0&0&0&0&-1\\
\hline
4&0&0&0&0&-5&0&0&0&0&1&1&0&0&0&0\\
0&0&4&0&0&0&0&-5&0&0&0&0&0&1&1&0\\
0&4&0&0&0&0&0&0&-5&0&0&0&0&1&1&0\\
0&0&0&4&0&0&0&0&0&-5&0&0&0&0&1&1\\
0&0&0&0&4&0&0&0&0&0&-5&0&0&0&0&1
\end{array}\right)$$

Dixon/Bézout type constructions

$$M = \begin{pmatrix}
0 & -4b & 0 & 4c-4b & 0 & 0 & 0 & 4b+c & 0 & a \\
0 & 0 & 0 & 4a & -4b & 0 & 4c-4b & a & 0 & 4b+c \\
-16c & 4b & 0 & 20b & 0 & 20c & 4a & -4b & 4a & 0 \\
0 & 0 & -16a & -16b & 0 & -16c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -16a & -16b & 0 & -16c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -16a & -16b & 0 & -16c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -16a & -16b & -16c \\
-16b & -16c & 0 & 0 & 0 & 20b & 0 & 20c & 4a & 4a \\
-16a & 0 & -16c & 0 & 4b & 20a & 20b & 0 & 20c & -4b \\
0 & -16a & -16b & 0 & -16c & 0 & 0 & 20a & 20b & 20c
\end{pmatrix}$$

$$\det M = E_M\, \texttt{Res}(f_0, f_1, f_2)$$

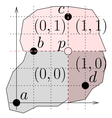
Since $\deg \texttt{Res}(f_0, f_1, f_2)$ is invariant under the presence of convex interior points, the inclusion of monomials corresponding to them in the polynomial system usually increases the degree of an extraneous factor E_M.

Matrix based methods are thus NOT "invariant" under the presence of convex interior points.

4 Support Hull

Informally, the **support hull** of a set of points $\mathcal{P}$ consists of

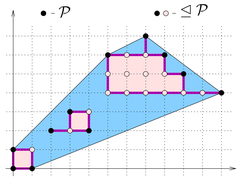
- the point set $\mathcal{P}$, and
- all other points for which there exist a point from $\mathcal{P}$ in every quadrant.



Points in different quadrants of point p.

The support hull of a set of points is thus

- "**inside**" of the convex hull of $\mathcal{P}$,
- rectilinearly defined,
- a set of lattice points,
- not necessarily a connected set.



EXAMPLE: The Support hull of support $\mathcal{P}$.

DEFINITION [*Quadrant relation*] Given $k \in \mathbb{Z}_2^d$, for points $p, q \in \mathbb{N}^d$, q is in the k^{th} octant of p,

$$p \overset{\leqslant}{}_k\, q \quad \Longleftrightarrow \quad \begin{cases} p_j \le q_j \ \text{if} \ k_j = 1, \\ p_j \ge q_j \ \text{if} \ k_j = 0, \end{cases} \quad \text{for} \quad j = 1, \ldots, d.$$

DEFINITION [**Support Hull**] Given a set of points $\mathcal{P} \subset \mathbb{N}^d$, define its support hull to be

$$\texttt{SupportHull}(\mathcal{P}) = \{\, p \mid \forall\, k \in \mathbb{Z}_2^d, \quad \exists\, q \in \mathcal{P}, \quad \text{such that} \quad p \overset{\leqslant}{}_k\, q \,\}.$$

For short, $p \trianglelefteq \mathcal{P}$ if $p \in \texttt{SupportHull}(\mathcal{P})$.

⋆ A *vertex* of a support hull is a lattice point in the support hull, whose removal will change the support hull.

5 Dixon resultant Formulation

Definition 5.1 (Dixon polynomial) *[Dix08, KSY94, Sax97] Given a polynomial system $\mathcal{F} = \{f_0, f_1, \ldots, f_d\}$, where $\mathcal{F} \subset \mathbb{Z}[x_1, \ldots, x_d][(c_{i,\alpha})]$, define its **Dixon polynomial** as*

$$\theta(f_0, \ldots, f_d) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} \pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\ \pi_1(f_0) & \pi_1(f_1) & \cdots & \pi_1(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(f_0) & \pi_d(f_1) & \cdots & \pi_d(f_d) \end{vmatrix}.$$

Hence $\theta(f_0, f_1, \ldots, f_d) \in \mathbb{Z}[x_1, \ldots, x_d, \overline{x}_1,\ ldots, \overline{x}_d][(c_{i,\alpha})]$, where $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_d$ are new variables.

Definition 5.2 (Dixon Matrix) *A Dixon polynomial $\theta(f_0, f_1, \ldots, f_d)$ can be written in bilinear form as*

$$\theta(f_0, f_1, \ldots, f_d) = \overline{X}^T \Theta X,$$

*where $\overline{X} = (\overline{\mathbf{x}}^{\beta_1}, \ldots, \overline{\mathbf{x}}^{\beta_k})$ and similarly, $X = (\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_l})$ are column vectors where X is an ordered set of monomials of $\theta(f_0, \ldots, f_d)$ in terms of variables $\{x_1, \ldots, x_d\}$ and $\overline{X}$ is an ordered set of monomials of $\theta(f_0, \ldots, f_d)$ in terms of variables $\{\overline{x}_1, \ldots, \overline{x}_d\}$. The $\Theta_{i,j}$ entry is the coefficient of monomial $\overline{\mathbf{x}}^{\beta_i}\mathbf{x}^{\alpha_j}$ in the polynomial $\theta(f_0, f_1, \ldots, f_d)$. The matrix Θ is called the **Dixon Matrix**.*

In most cases, the determinant of any maximal minor of the Dixon matrix is a multiple of the (toric) resultant of the given polynomial system.

6 Main Result

For an unmixed generic polynomial system, the inclusion of monomials corresponding to support hull interior point does not change the size of the Dixon matrix, and hence, the degree of its (maximal minor) determinant .

Theorem 6.1 (Main) *For an unmixed generic polynomial system with support $\mathcal{A} = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_d \rangle$, where $\mathcal{A} = \mathcal{A}_i = \mathcal{A}_j$, and a point $p \in \mathbb{N}^d$,*

$$p \trianglelefteq \mathcal{A} \quad \Longleftrightarrow \quad |\theta_{\mathcal{A}\cup\{p\}}| = |\theta_{\mathcal{A}}|,$$

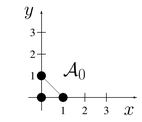
that is, a point is in the support hull if and only if it does not change the structure of the Dixon polynomial.

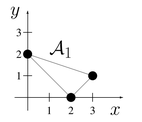
1. The support hull thus determines the size of the Dixon/Bézout-based resultant matrix.
2. Polynomials systems with supports whose *support hull vertices* are not the same as convex hull vertices, will have extraneous factors in the determinant of the Dixon matrix (or its maximal minor).

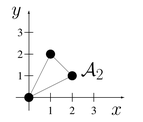
7 Application: Mixed Systems - Support Overlap

Consider a support translation $\mathcal{A} + t = \langle \mathcal{A}_0 + t_0, \mathcal{A}_1 + t_1, \ldots, \mathcal{A}_d + t_d \rangle$, where $t_i = (t_{i,1}, \ldots, t_{i,d})$, which amounts to multiplying individual polynomials f_i by $\mathbf{x}^{t_i}$.

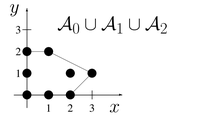
Given a polynomial system with support:



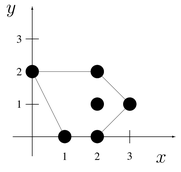




- The mixed volume is $\langle 8, 3, 4 \rangle = 15$
- The Dixon dialytic matrix is of size 18×18, with an extraneous factor of degree
  - 1 in coefficients of f_0,
  - 1 in the coefficients of f_1,
  - 1 in the coefficients of f_2.



If $t = \langle (2,1), (0,0), (1,0) \rangle$ is used, then the support associated with $\mathcal{A}+t$ has a smaller overall support hull than that of $\mathcal{A}$.



$(\mathcal{A}_0 + t_0) \cup (\mathcal{A}_1 + t_1) \cup (\mathcal{A}_2 + t_2)$

[Cht03] The Dixon dialytic matrix of the translated system is of size 15×15, with extraneous factors of degree

- 0 in coefficients of f_0,
- 0 in the coefficients of f_1,
- 0 in the coefficients of f_2,

i.e., its determinant is the toric resultant.

- The above optimization also works for Dixon matrices; in this case, the 9×9 Dixon matrix is reduced to 8×8.
- Smaller Dixon/Bézout-based resultant matrices can be constructed if their support hull can be made "smaller" by translation.
- For the Dixon dialytic construction, set of d supports have to be considered, instead of $d+1$ to achieve optimal results.
- Newton sparse resultant construction is insensitive to support position, and hence the above optimization is not applicable.

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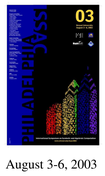

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