Conditions for Exact Resultants using the Dixon Formulation

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ABSTRACT

A structural criteria on polynomial systems is developed for which the generalized Dixon formulation of multivariate resultants defined by Kapur, Saxena and Yang (1994) computes the resultant exactly. The concept of a Dixon-exact support (the set of exponent vectors of terms appearing in a polynomial system) is introduced so that the Dixon formulation produces the exact resultant for generic unmixed polynomial systems whose support is Dixon-exact. A geometric operation, called direct-sum, on the supports is defined that preserves the property of supports being Dixon-exact. Generic n-degree systems and multigraded systems are shown to be a special case of generic unmixed polynomial systems whose support is Dixon-exact. Using a scaling techniques discussed by Kapur and Saxena (1997), a wide class of polynomial systems can be identified for which the Dixon formulation produces exact resultants. This analysis can be used to classify terms appearing in the convex hull (also called the Newton polytope) of the support of a polynomial system that can cause extraneous factors in the computation of a projection operation by the generalized Dixon formulation. For the bivariate case, a complete analysis of the terms corresponding to the exponent vectors in the Newton polytope of the support of a polynomial system is given vis a vis their role in producing extraneous factors in a projection operator. A necessary and sufficient condition is developed for a support to be Dixon-exact. Such an analysis is likely to give insights for the general case of elimination of arbitrarily many variables.

1. INTRODUCTION

A structural criteria on multivariate polynomial systems is developed such that the generalized Dixon formulation of multivariate resultants [7, 13] as well as the associated sparse resultant construction recently obtained by the authors from this formulation [5] computes the resultant exactly, i.e., these constructions do not produce any extraneous factors in the resultant computations for such polynomial systems. This result is of considerable practical significance. Extraneous factors arising in resultant computations of polynomial systems is a key problem faced when any multivariate resultant method for simultaneously eliminating many variables is used for elimination in a variety of applications including computer vision, robotics and kinematics, control theory, solid and geometric modeling, geometry theorem proving, biology, etc. A long-term goal of this research is to identify subsets of terms in relation to the supports (sets of exponent vectors of terms) of polynomials in a polynomial system that can produce extraneous factors in computing a *projection operator* (which is a nonzero multiple of the resultant) using the multivariate generalized Dixon resultant formulation.

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Towards this objective, the notion of a *Dixon-exact* support is introduced to identify generic unmixed polynomial systems for which the Dixon resultant formulation produces the exact resultant. A geometric operation, called *direct-sum*, on supports is defined that preserves the property of supports being Dixon-exact. Generic n-degree systems for which the Dixon formulation is known to compute exact resultants [7, 14] are shown to be a special case of generic unmixed polynomial systems whose support is Dixon-exact. This is discussed in Section 3, after Section 2 which gives the background on supports, the Dixon resultant formulation, Dixon polynomials and Dixon matrices.

Generic multigraded systems introduced by Strumfels and Zelvinsky [16] for which they gave a Sylvester type formula for sparse resultants are also shown to be a special case of generic unmixed polynomial systems whose support is Dixon-exact. This insight gives a new result that the generalized Dixon formulation can be used to compute exact resultants for generic multigraded polynomial systems. Besides these systems, other generic unmixed multihomogeneous polynomial systems with Dixon-exact supports are identified for which exact resultants can be computed, thus extending results in [7, 16, 14].

Generic unmixed polynomial systems with different supports are shown to exist such that the Newton polytope (the convex hull) of their supports is the same, but for some supports, the generalized Dixon resultant formulation produces exact resultants whereas for others, it generates extraneous factors. That is despite the fact that the degree of the resultants of these polynomial systems is the same, since it is determined by the Newton polytope. Terms corresponding to the exponent vectors in a Newton polytope can be classified based on whether they are likelty to contribute extraneous factor in resultant computations. This is discussed in Section 4.

For the bivariate case, a complete analysis of terms in a polynomial system is provided, vis a vis their role in producing extraneous factors in a projection operator computed by the generalized Dixon formulation. A complete chararcterization (i.e. a necessary and sufficient condition) of the support of a polynomial system is given for which exact resultant can be computed. In other cases, the degree of extraneous factors in a projection operator can be determined using a technique based on successive partitioning of the support of a polynomial system. Such an analysis is likely to give insights for the general case of elimination of arbitrarily many variables. This is discussed in Section 5.

Using the techniques discussed in [12], a wide class of polynomial systems can be identified for which the Dixon formulation produces exact resultants. It is proved in [12] that for a class of polynomial systems, the projection operator computed by the generalized Dixon formulation can be related to the projection operator computed by the same method for a simpler polynomial system (with polynomials whose support is a smaller Newton polytope and of lower degrees). This result can be exploited in many different ways. Firstly, the projection operator for such smaller polynomial systems can be computed much faster and using less memory space, than similar computations for the larger systems. Secondly, extraneous factors in the projection operator can be shown to be powers of the extraneous factors in the projection operator of the related smaller polynomial system. In other words, if the generalized Dixon formulation does not generate extraneous factors in the projection operator of the smaller system, then it is guaranteed not to do so for the larger one either.

The problem of extraneous factors is not peculiar to the elimination method based on the generalized Dixon formulation. In fact, none of the resultant based elimination methods – the Macaulay formulation, Dixon formulation or the sparse resultant formulation, compute the exact resultant of arbitrary nonhomogeneous polynomial systems. Instead, these methods compute various multiples of the resultant, known as projection operators, which may contain extraneous factors besides the resultant. Since the information about the solutions of a polynomial system is completely contained in its resultant, the extraneous factors in a projection operator do not offer any additional information. Instead, they make it more difficult to identify the resultant in a projection operator, as for each factor in a projection operator, it must be checked whether it is extraneous or if it is a part of the resultant, and this check can be resource-consuming. The presence of extraneous factors can also make computing a projection operator impractical, since extraneous factors can increase the total degree of the projection operator considerably, and the computational complexity of the projection operator using the generalized Dixon method (as well as other resultant formulations) is determined by its degree.

The results reported in this paper extend our earlier results about the generalized Dixon formulation, a method for simultaneously eliminating many variables for a large class of polynomial systems and computing a projection operator from which the resultant can be extracted [13]. This method has been experimentally found to be superior in performance on a wide variety of examples, in comparison with other elimination methods including Macaulay resultants, sparse resultants [3, 15, 9], the characteristic set construction [4], and the Gröbner basis construction [1, 2]. The method takes less time, less space, as well as the extraneous factors seem to be fewer (except in the case of the Gröbner basis method which gives the exact resultant) [10]. We also proved that for the unmixed case, the Dixon formulation, in fact, *implicitly* exploits the sparse structure of the polynomial system, i.e., its computational complexity is governed by the Newton polytope of the unmixed system, not by the Bezout bound as is the case for Macaulay resultants [11]. Recently in [5], we have given a simple algorithm for efficiently generating sparse resultant matrices using the Dixon formulation, in contrast to other known techniques for generating sparse resultant matrices based on explicitly exploiting the support of the polynomial system.

2. DEFINITIONS & NOTATION

DEFINITION 2.1. *Given a polynomial* $f = c_{\alpha_1}\mathbf{x}^{\alpha_1} + \cdots + c_{\alpha_n}\mathbf{x}^{\alpha_n} \in \mathbb{C}[x_1, \ldots, x_d]$ *and* $\mathbf{x}^{\alpha_i} = x_1^{\alpha_{i,1}} x_2^{\alpha_{i,2}} \ldots x_d^{\alpha_{i,d}}$, *let the support of* f *be the set* $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$, *where* $\alpha_i \in \mathbb{N}^d$ *and* $c_{\alpha_i} \neq 0$. *Define a* ***support map*** $\mathsf{S}_{\mathbf{x}} : \mathbb{C}[x_1, \ldots, x_d] \rightarrow \mathbb{N}^d$, *and* $\mathsf{S}_{\mathbf{x}}(f) = \mathcal{A}$, *where* $\mathbf{x} = [x_1, \ldots, x_d]$.

DEFINITION 2.2. *Define* $\pi_i(\mathbf{x}^\alpha) = \overline{x}_1^{\alpha_1} \cdots \overline{x}_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} \cdots x_d^{\alpha_d}$, *where* $i \in \{0, \ldots, d\}$ *and* $\overline{x}_i$*'s are new variables;* $\pi_0(\mathbf{x}^\alpha) = \mathbf{x}^\alpha$. π_i *is extended to polynomials as:*

$$\pi_i(f(x_1, \ldots, x_d)) = f(\overline{x}_1, \ldots, \overline{x}_i, x_{i+1}, \ldots, x_d).$$

DEFINITION 2.3. *Given* $P = \{f_0, f_1, \ldots, f_d\}$, *where* $P \subset \mathbb{C}[x_1, \ldots, x_d]$, *define its* ***Dixon polynomial*** *as*

$$\theta(f_0, \ldots, f_d) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} \pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\ \pi_1(f_0) & \pi_1(f_1) & \cdots & \pi_1(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(f_0) & \pi_d(f_1) & \cdots & \pi_d(f_d) \end{vmatrix}.$$

Hence $\theta(f_0, f_1, \ldots, f_d) \in \mathbb{C}[x_1, \ldots, x_d, \overline{x}_1, \ldots, \overline{x}_d]$, *where* $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_d$ *are new variables.*

The order in which original variables in $\mathbf{x}$ are replaced by new variables in $\overline{\mathbf{x}}$ is significant in the sense that Dixon polynomials computed using two different orderings may be different. This is discussed and illustrated later.

A polynomial system $P = \{f_0, \ldots, f_d\}$ is called *unmixed* if each polynomial has the same support, i.e., for all i, $\mathsf{S}_{\mathbf{x}}(f_i) = \mathcal{A}$. For unmixed generic systems, we will use $\theta_{\mathcal{A}} = \theta(f_0, f_1, \ldots, f_d)$, to stress its dependence on the support $\mathcal{A}$ of the polynomial system.

DEFINITION 2.4. *A Dixon polynomial $\theta(f_0, \ldots, f_d)$ can be written in bilinear form as*

$$\theta(f_0, f_1, \ldots, f_d) = \overline{X}\Theta X^T,$$

where $\overline{X} = [\overline{\mathbf{x}}^{\beta_1}, \ldots, \overline{\mathbf{x}}^{\beta_k}]$ and similarly, $X = [\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_l}]$ are row vectors with $\alpha_i \in \mathsf{S}_{\mathbf{x}}(\theta)$ and $\beta_i \in \mathsf{S}_{\overline{\mathbf{x}}}(\theta)$. The $k \times l$ matrix Θ is called the ***Dixon Matrix****.*

As proved in [11, 14], the determinant of a maximal minor of the Dixon matrix Θ is, in general, a *projection operator*, a non-trivial multiple of the resultant, provided that the Dixon matrix Θ has an independent column. Each entry in Θ is a polynomial in the coefficients of the polynomials in P, and its degree in the coefficients of any single polynomial is at most 1. Hence, the projection operator computed using the Dixon formulation can be at most of degree $|X|$ in the coefficients of any single polynomial.

It is known that the degree of the toric resultant of a *generic unmixed* polynomial system with support $\mathcal{A}$ is $d!\mathsf{Vol}_d(N(\mathcal{A}))$ in the coefficients of any polynomial [6], where $N(\mathcal{A})$ is the Newton polytope (the convex hull) of $\mathcal{A}$ and Vol_d gives the Euclidean volume of a Newton polytope.

$|X|$ reveals if any extraneous factor exists in the determinant of a maximal minor, and if so, gives a lower bound on the degree of the extraneous factor. Since $|X| = |\mathsf{S}_{\mathbf{x}}(\theta)|$, we are interested in estimating $|\mathsf{S}_{\mathbf{x}}(\theta)|$. The support of the Dixon polynomial can also be analyzed in terms of $\overline{x}_i$ variables, but this is equivalent to doing the analysis in terms of x_i variables if the variable order is completely reversed.

PROPOSITION 2.1. *Given an unmixed generic polynomial system $P = \{f_0, f_1, \ldots, f_d\}$ with a support $\mathcal{A}$,*

$$|\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}})| \geq d!\mathsf{Vol}_d(N(\mathcal{A})).$$

PROOF. : If the resultant of P exists, then there exist a maximal minor in its Dixon matrix whose determinant is a non-zero projection operator (see [11], [8]). Since the degree of the resultant is $d!\mathsf{Vol}_d(N(\mathcal{A}))$ in the coefficients of any polynomial in P, the size of the Dixon matrix has to be at least that big, and hence, the size of the support of the Dixon polynomial. □

The Dixon polynomial of a polynomial system P can be decomposed into a sum of smaller Dixon polynomials of polynomial systems with $d+1$ monomials. This nice identity for the Dixon polynomial is used later in proofs.

PROPOSITION 2.2. *Let $P = \{f_0, f_1, \ldots, f_d\}$ be a polynomial system and let $\mathcal{A} = \cup_{i=0}^{d}\mathsf{S}_{\mathbf{x}}(f_i)$. For $\sigma \subseteq \mathcal{A}$, where $\sigma = \{\sigma_0, \ldots, \sigma_d\}$, let $\sigma(\mathbf{c})$ stand for the matrix whose $(i,j)^{th}$ entry is c_{i,σ_j}, where c_{i,σ_j} is the coefficient of monomial $\mathbf{x}^{\sigma_j}$ in f_i; similarly, let $\sigma(\mathbf{x})$ be the matrix whose $(i,j)^{th}$ entry is $\pi_i(\mathbf{x}^{\sigma_j})$. Then*

$$\theta(f_0, f_1, \ldots, f_d) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \sum_{\substack{\sigma \subseteq \mathcal{A} \\ |\sigma| = d+1}} |\sigma(\mathbf{c})||\sigma(\mathbf{x})|.$$

PROOF. : Let $f_i = \sum_{j=1}^{n} c_{i,\alpha_j}\mathbf{x}^{\alpha_j}$, (assuming 0 for coefficients if necessary). The Dixon polynomial of P is given by

$$\begin{aligned}
\theta(f_0, \ldots, f_d) &= \\
&= \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \left| \begin{pmatrix} \pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\ \pi_1(f_0) & \pi_1(f_1) & \cdots & \pi_1(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(f_0) & \pi_d(f_1) & \cdots & \pi_d(f_d) \end{pmatrix}^T \right| \\
&= \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \left| \begin{pmatrix} c_{0,\alpha_1} & c_{0,\alpha_2} & \cdots & c_{0,\alpha_n} \\ c_{1,\alpha_1} & c_{1,\alpha_2} & \cdots & c_{1,\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d,\alpha_1} & c_{d,\alpha_2} & \cdots & c_{d,\alpha_n} \end{pmatrix} \begin{pmatrix} \pi_0(\mathbf{x}^{\alpha_1}) & \pi_1(\mathbf{x}^{\alpha_1}) & \cdots & \pi_d(\mathbf{x}^{\alpha_1}) \\ \pi_0(\mathbf{x}^{\alpha_2}) & \pi_1(\mathbf{x}^{\alpha_2}) & \cdots & \pi_d(\mathbf{x}^{\alpha_2}) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0(\mathbf{x}^{\alpha_n}) & \pi_1(\mathbf{x}^{\alpha_n}) & \cdots & \pi_d(\mathbf{x}^{\alpha_n}) \end{pmatrix} \right| \\
&= \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \sum_{\substack{\sigma \subseteq \mathcal{A} \\ |\sigma| = d+1}} |\sigma(\mathbf{c})||\sigma(\mathbf{x})|
\end{aligned}$$

(by Cauchy-Binet Formula).□

The above identity shows that if generic coefficients are assumed in the polynomial system, then the support of the Dixon polynomial depends entirely on the support of the polynomial system.

3. DIXON-EXACT SUPPORTS

DEFINITION 3.1. *Given a generic unmixed polynomial system P with support $\mathcal{A}$, $\mathcal{A}$ is called Dixon-exact if there exists a variable order on $\mathbf{x}$ resulting in the Dixon polynomial such that*

$$|\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}})| = d!\mathsf{Vol}_d(N(\mathcal{A})).$$

DEFINITION 3.2. *A set $\rho \subset \mathbb{N}^d$ is called a basis simplex if*

- $|\rho| = d+1$, *and*
- *for all $p_i \in \rho$, $p_{i,j} = 0$, $1 \leq j \leq d$, except possibly one $p_{i,k} \neq 0$.*

If $\mathsf{Vol}_d(N(\rho)) > 0$, then there exist d points in ρ such that each is lying on a unique coordinate axis, and the $d+1$-th point is on any axis.

For ρ to have a zero d-dimensional volume, it must be of dimension less than d, i.e., there must exist a coordinate, say i-th, such that for all points in ρ, their i-th coordinate is 0. In terms of monomials appearing in a polynomial system, this implies that some variable does not appear at all, and hence, two rows in the expansion of the determinant for the Dixon polynomial will be the same. That is, if $\mathsf{Vol}_d(N(\rho)) = 0$, then $|\mathsf{S}_{\mathbf{x}}(\theta_\rho)| = 0$, because the Dixon polynomial is zero.

Assume that $\rho = \{p_0, \ldots, p_d\}$, and $p_1, \ldots, p_d$ are points lying on respective axes $x_1, \ldots, x_d$. Let $|p|$ stand for the length of vector p, i.e., the distance of point p from the origin. Assume that the first point p_0 is lying on the same axis as p_j, but $|p_0| < |p_j|$, i.e., p_0 is closer to the origin than p_j. Then,

$$d!\mathsf{Vol}_d(N(\rho)) = |p_j - p_0| \prod_{\substack{i=1\\ i\neq j}}^{d} |p_i|.$$

PROPOSITION 3.1. *For a generic unmixed polynomial system with support ρ that is a basis simplex,*

$$|\mathsf{S_x}(\theta_\rho)| = d!\mathsf{Vol}_d(N(\rho)),$$

i.e. ρ is Dixon-exact with respect to any variable order.

PROOF. : Assume $\rho = \{p_0, \ldots, p_d\}$ as above. The expression for the Dixon polynomial of such a generic system is given by

$$D = \begin{vmatrix} x_j^{p_0} & x_1^{p_1} & x_2^{p_2} & \cdots & x_d^{p_d} \\ x_j^{p_0} & \overline{x}_1^{p_1} & x_2^{p_2} & \cdots & x_d^{p_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_j^{p_0} & \overline{x}_1^{p_1} & \overline{x}_2^{p_2} & \cdots & x_d^{p_d} \\ \overline{x}_j^{p_0} & \overline{x}_1^{p_1} & \overline{x}_2^{p_2} & \cdots & x_d^{p_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{x}_j^{p_0} & \overline{x}_1^{p_1} & \overline{x}_2^{p_2} & \cdots & \overline{x}_d^{p_d} \end{vmatrix} =$$

$$x_j^{p_0}\overline{x}_j^{p_0}(x_j^{p_j-p_0} - \overline{x}_j^{p_j-p_0}) \prod_{\substack{i=1\\ i\neq j}}^{d} (x_i^{p_i} - \overline{x}_i^{p_i}).$$

The Dixon polynomial is then $CD\prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i}$, where C is the coefficient matrix. We thus have

$$\mathsf{S_x}(\theta_\rho) = \{\, \alpha \;|0 \le \alpha_i < p_i \text{ for } j \ne i = 1, \ldots, d \text{ and } p_0 \le \alpha_j < p_j \}.$$

Thus, the result follows. □

It is easy to see that $|\rho + \mathsf{S_x}(\theta_\rho)| = (d+1)|\mathsf{S_x}(\theta_\rho)|$. The above relation is used in Proposition 3.3 to follow.

PROPOSITION 3.2. *Let ρ_1 and ρ_2 be two basis simplexes, such that for every $p \in \rho_1$ there exist $s, t \in \rho_2$, such that $s_i \le p_i \le t_i$ for all $i = 1, \ldots, d$, then*

$$\mathsf{S_x}(\theta_{\rho_1}) \subseteq \mathsf{S_x}(\theta_{\rho_2}).$$

This follows from the proof of Proposition 3.1, and the fact that a Dixon polynomial is a sum of determinants corresponding to the basis simplexes as per Proposition 2.2.

If the support of a polynomial system is $\mathcal{A} = \rho_1 \cup \rho_2$ where ρ_1, ρ_2 satisfy the conditions in the above proposition, then, $\mathsf{S_x}(\theta_{\rho_1\cup\rho_2}) = \mathsf{S_x}(\theta_{\rho_2})$.

DEFINITION 3.3. *A support $\mathcal{A}$ is called a basis support if any $d+1$ subset of it is a basis simplex, of dimension d or less.*

THEOREM 3.1. *Given a generic, unmixed polynomial system P with a basis support $\mathcal{A}$, $\mathcal{A}$ is Dixon-exact.*

The proof follows from Proposition 2.2.

Note that the one-dimensional case is a special case of the above theorem. The Dixon resultant formulation yields exact resultant in the univariate case when the degrees of the two polynomials are the same (the unmixed case).

DEFINITION 3.4. *Let P and Q be supports.*

- *The **Minkowski Sum** of $\mathcal{P}$ and $\mathcal{Q}$, denoted by $\mathcal{P}+\mathcal{Q}$, is $\mathcal{P}+\mathcal{Q} = \{p+q | p \in \mathcal{P} \text{ and } q \in \mathcal{Q}\}$, where $p+q$ is the regular vector sum.*

  We also use the notation $p + \mathcal{Q}$ to stand for $\{p\} + \mathcal{Q}$.
- *For any non-negative integer k, let*

  $$k\mathcal{P} = \{\, kp = (kp_1, \ldots, kp_d) \,|p = (p_1, \ldots, p_d) \in \mathcal{P}\}.$$

Below, we define a geometric operation on supports that preserves the property of a support being Dixon-exact. By iterating this geometric operation on supports, we can obtain other Dixon-exact supports.

DEFINITION 3.5. *Given two supports $\mathcal{P}$ and $\mathcal{Q}$, define direct sum of $\mathcal{P}$ and $\mathcal{Q}$ to be*

$$\mathcal{P} \oplus \mathcal{Q} = \{(p_1, \ldots, p_k, q_1, \ldots, q_l)|\; p = (p_1, \ldots, p_k) \in \mathcal{P} \subset \mathbb{N}^k, \text{ and } \; q = (q_1, \ldots, q_l) \in \mathcal{Q} \subset \mathbb{N}^l, \}$$

where $k + l = d$.

Note that $\mathcal{P} \oplus \mathcal{Q} \subset \mathbb{N}^d$, and $\mathsf{Vol}_d(N(\mathcal{P}\oplus\mathcal{Q})) = \mathsf{Vol}_k(N(\mathcal{P}))\ \mathsf{Vol}_l(N(\mathcal{Q}))$. The direct sum can be thought as the Minkowski sum where $\mathcal{P}$ and $\mathcal{Q}$ are embedded into $\mathbb{N}^d$, and added.

PROPOSITION 3.3. *Suppose $\mathcal{A} = \mathcal{P} \oplus \mathcal{Q}$, where $\mathcal{P}, \mathcal{Q}$ are Dixon-exact supports using the variable orders $X_\mathcal{P}$ and $X_\mathcal{Q}$, respectively. If $\mathcal{Q}$ is a basis support, then $\mathcal{A}$ is Dixon-exact with respect to the variable order $\{X_\mathcal{P}, X_\mathcal{Q}\}$.*

PROOF. : Using the variable order $\{X_\mathcal{P}, X_\mathcal{Q}\}$, the Dixon polynomial for the polynomial system with support $\mathcal{A}$ is:

$$\theta(f_0, \ldots, f_d) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} \pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{k-1}(f_0) & \pi_{k-1}(f_1) & \cdots & \pi_{k-1}(f_d) \\ \hline \pi_k(f_0) & \pi_k(f_1) & \cdots & \pi_k(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{k+l-1}(f_0) & \pi_{k+l-1}(f_1) & \cdots & \pi_{k+l-1}(f_d) \\ \hline \pi_{k+l}(f_0) & \pi_{k+l}(f_1) & \cdots & \pi_{k+l}(f_d) \end{vmatrix}.$$

The above matrix has been partitioned into 3 sets of rows. In the first part, only variables in $X_\mathcal{P}$ are replaced with variables in $X_\mathcal{Q}$ not getting changed; in the second part $X_\mathcal{Q}$ variables are changed with all variables in $X_\mathcal{P}$ already replaced by new variables. In the last row, all variables have been replaced by new variables.

Since the last row does not contain any original variable, it will not contribute to the support of the Dixon polynomial.

Using the Laplace formula, the above determinant can be written as the sum of products of the determinant of a minor from the upper part, the determinant of a minor with a different subset of columns from the second part, and the element in the column not considered, from the last row.

The minor from the upper part will have support which is contained in

$$\mathsf{S_x}(\theta_\mathcal{P}) + \underbrace{\mathcal{Q} + \cdots + \mathcal{Q}}_{k},$$

and any minor from the lower part will have support $\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{Q}})$. Hence the support of the Dixon polynomial is contained in the Newton polytope of the Minkowski sum of

$$\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{P}}) + \underbrace{\mathcal{Q} + \cdots + \mathcal{Q}}_{k} + \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{Q}}).$$

Since $\mathcal{P}$ and $\mathcal{Q}$ are supports based on different variables,

$$|\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}})| \;=\; |\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{P}})||\underbrace{\mathcal{Q} + \cdots + \mathcal{Q}}_{k} + \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{Q}})|.$$

Let $\mathcal{Q}'$ be a basis simplex in $\mathcal{Q}$ with maximum volume.

$$\text{For} \qquad p,\ q \in \underbrace{\mathcal{Q}' + \cdots + \mathcal{Q}'}_{k}$$
$$p + \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{Q}}) \cap q + \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{Q}}) = \oslash,$$

because the maximum of any point coordinate in $\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{Q}})$ is smaller than the corresponding maximum in $\mathcal{Q}'$. Then,

$$|\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}})| = |\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{P}})||\underbrace{\mathcal{Q}' + \cdots + \mathcal{Q}'}_{k}||\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{Q}})| =$$

$$\binom{k+l}{k} k!\mathsf{Vol}_d(N(\mathcal{P}))l!\mathsf{Vol}_d(N(\mathcal{Q})) = d!\mathsf{Vol}_d(N(\mathcal{A})).$$

This implies that the support $\mathcal{A}$ is Dixon-exact. □

In the above proof, the following property is used:

PROPOSITION 3.4. *Let $\mathcal{Q}$ be a support that is a basis simplex, then* $|\underbrace{\mathcal{Q} + \cdots + \mathcal{Q}}_{k}| = \binom{d+k}{k}$.

PROOF. : By induction on the dimension d. The basis case is $d = 1$. Then $\mathcal{Q}$ is just 2 points on a line, and

$$|\underbrace{\mathcal{Q} + \cdots + \mathcal{Q}}_{k}| = k + 1.$$

Assume that the proposition is true for d. $\mathcal{Q} = \mathcal{Q}' \cup \{p\}$ where $\mathcal{Q}'$ has d points, where all have one coordinate value 0, and p has non-zero value in that coordinate. Let

$$S_j = j\{p\} + \underbrace{\mathcal{Q}' + \cdots + \mathcal{Q}'}_{k-j} \quad \text{for} \quad j = 0, \ldots, k.$$

Note that $\mathcal{Q} + \cdots + \mathcal{Q} = \cup_{j=0}^{k} S_j$. Also note that $S_j \cap S_i = \oslash$ for $j \neq i$. Using the induction hypothesis,

$$|\underbrace{\mathcal{Q} + \cdots + \mathcal{Q}}_{k}| \;=\; \sum_{j=0}^{k} \binom{d+k-j}{k-j} = \sum_{i=0}^{k} \binom{d+i}{i}$$
$$=\; \binom{d+1+k}{k}. \square$$

THEOREM 3.2. *Given a generic unmixed polynomial system whose support is a direct sum of k Dixon-exact supports $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k$, where $\mathcal{A}_i$, for $1 \leq i \leq k$, is a basis support, the Dixon formulation computes the exact resultant.*

The proof follows by induction from Proposition 3.3.

3.1 n-Degree Systems

DEFINITION 3.6. *A support $\mathcal{A}$ is n-degree if there exists nonnegative integers $k_1, \ldots, k_d$ such that $\mathcal{A} = \{p \mid 0 \leq p_i \leq k_i\}$.*

PROPOSITION 3.5. *A n-degree support $\mathcal{A}$ can be expressed as $\mathcal{A} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$, where $\mathcal{X}_i$ is 1-dimensional basis support and each $p \in \mathcal{X}_i$ satisfies $0 \leq p \leq k_i$.*

For generic unmixed n-degree polynomial systems, the Dixon formulation has been shown to compute exact resultants (see [14]). Here we show that this result is a special case of Theorem 3.2.

COROLLARY 3.5.1. *The Dixon resultant formulation yields the exact resultant for a generic unmixed n-degree polynomial system.*

PROOF. : A generic unmixed n-degree polynomial system has an n-degree support, which by the above proposition, can be expressed as a direct sum of Dixon-exact supports. The statement follows from Theorem 3.2. □

4. MULTIHOMOGENEOUS SYSTEMS

DEFINITION 4.1. *A polynomial f is called multihomogeneous of type $(l_1, l_2, \ldots, l_r; k_1, k_2, \ldots, k_r)$ if*

$$f = \sum c_\alpha \mathbf{x}_1^{k_1} \mathbf{x}_2^{k_2} \cdots \mathbf{x}_r^{k_r}, \ \textit{where}\ \mathbf{x}_i^{k_i} = x_{i,1}^{p_1} \cdots x_{i,l_i+1}^{p_{l_i+1}},$$
$$\textit{and} \qquad p_1 + p_2 + \cdots p_{l_i} + p_{l_i+1} = k_i.$$

That is, variables are partitioned into r blocks of size $l_i, 1 \leq i \leq r$, and the polynomial is made homogeneous using a unique homogenizing variable x_{i,l_i+1} for each block $\mathbf{x}_i$, and its degree in the variables in the block $\mathbf{x}_i$ is k_i.

A multihomogeneous polynomial of type $(l_1, l_2, \ldots, l_r;$ $k_1, k_2, \ldots, k_r)$ is *full* if every term (after homogenization) of degree k_i in block $\mathbf{x}_i$, $1 \leq i \leq r$, has a nonzero coefficient in the polynomial.

A polynomial is called multihomogeneous of some type, if it can be multi-homogenized into such a multihomogeneous polynomial. Note that a polynomial can be made multihomogeneous in a number of ways, depending on the partition of variables. We will call the support of a multihomogeneous polynomial to be multihomogeneous as well; further, the support of a full multihomogenous polynomial is called full.

PROPOSITION 4.1. *A full, multihomogeneous support $\mathcal{A}$ of type $(l_1, l_2, \ldots, l_r; k_1, k_2, \ldots, k_r)$ can be written as*

$$\mathcal{A} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_r,$$

where $\mathcal{X}_i$ is the support of a dense polynomial of total degree of k_i, in the variables of the i^{th} block $\mathbf{x}_i$.

4.1 Multigraded Systems

DEFINITION 4.2. *A multihomogeneous system of type $(l_1, l_2, \ldots, l_r; k_1, k_2, \ldots, k_r)$ is called* ***multigraded*** *if $\forall i = 1, \ldots, r$ either $l_i = 1$ or $k_i = 1$.*

Multigraded systems were introduced in [16] as a special case of multihomogeneous systems.

Multihomogeneous supports are direct sum of smaller supports as shown above, yet they are not necessarily Dixon-exact. But multigraded systems are.

PROPOSITION 4.2. *A full multigraded support $\mathcal{A}$ can be expressed as $\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_r$, where $\mathcal{X}_i$ is Dixon-exact for all $i = 1, \ldots, r$.*

PROOF. : When $l_i = 1$ for some $1 \leq i \leq r$, $\mathcal{X}_i$ is one dimensional support and hence is Dixon-exact.

Whenever $k_i = 1$ for some $1 \leq i \leq r$, $\mathcal{X}_i$ consists of a zero l_i-tuple, and l_i-tuples in which exactly one coordinate is non-zero, and has value 1. $\mathcal{X}_i$ is then a basis simplex, which is Dixon-exact. □

From the above Proposition and Theorem 3.2, the following new result is obtained about the Dixon resultant formulation, analogous to a similar result about sparse resultants about multigraded systems given in [16].

COROLLARY 4.2.1. *The Dixon resultant formulation yields exact resultant for a generic unmixed multigraded polynomial system.*

4.2 Dixon-exact Multihomogeneous Systems

Not all Dixon-exact supports are multigraded. There are multihomogeneous systems which are not multigraded but whose support is still Dixon-exact. For example, consider an unmixed generic polynomial system where each polynomial has $\{1, y^2, z, x^3, x^3y^2, x^3z, x^2y\}$ as terms. Clearly this is not a multigraded polynomial system, yet it is Dixon-exact since its support is the direct sum of exponent vectors of $1, y^2, z$ and exponent vectors of $1, x^3$, where the exponent vector corresponding to x^2y is Dixon-interior.

Theorem 3.2 in the previous section is, thus, a strong generalization of the theorem in [16] in the sense that there are polynomial systems of which multigraded systems are special cases, for which a Sylvester-type resultant formula can be given using the Dixon resultant formulation.

As pointed out in [16], there are $r!$ resultant matrices for multigraded systems whose determinant is the resultant. It can be shown that the Dixon resultant formulation can construct any of $r!$ exact matrices as well, depending on the order of given blocks used in the construction. The order within blocks will change only the monomial multipliers.

Figure 1 depicts the support of a multigraded system of type $(1, 2; 2, 1)$, where variable blocks are $\{x\}$ and $\{y, z\}$. Under variable orders $[x, y, z], [x, z, y], [y, z, x]$ and $[z, y, x]$ Dixon resultant formulation yields 6×6 Dixon matrix, from which exact resultant is extracted in accordance with Theorem 3.2.

The sensitivity of the Dixon resultant computation to the variable ordering is reflected for multigraded systems. Particularly, if the variable ordering used does not coincide with the block structure, then the Dixon formulation can lead to extraneous factors in the resultant computation. For the above example, if variable order $[y, x, z]$ or $[z, x, y]$ is used, they give rise to Dixon matrix of size 8×8, resulting in an extra factor of degree 2 in the coefficients of any one polynomial in the polynomial system.

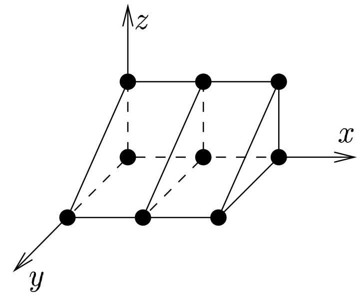

Figure 1: Multigraded system of type $(1, 2; 2, 1)$

4.3 Other Terms in Newton Polytope

DEFINITION 4.3. *A point $p \in \mathcal{A}$ is called Dixon-interior if*

$$\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}}) = \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}-\{p\}}).$$

In the literature, a point is called interior with respect to a support $\mathcal{A}$ if it belongs to the convex hull of $\mathcal{A}$. It is known that the presence of monomials corresponding to points not in the support of a polynomial system but inside the convex hull do not change the degree of its resultant. Yet there exist examples where given an unmixed polynomial system with support $\mathcal{A}$, and point p not in $\mathcal{A}$ but in its convex hull, the Dixon matrix of $\mathcal{A} \cup \{p\}$ is bigger than that of $\mathcal{A}$, resulting in the projection operator containing extraneous factors. This is the rationale for introducing the above definition which is much more restricted.

For example, in the bivariate case, a polynomial system with support $\{(0, 0), (2, 0), (0, 2)\}$ results in the Dixon matrix of size 4×4. If a monomial corresponding to the point $(1, 1)$ not in the support but in its convex hull is added to the polynomial system, the Dixon matrix becomes of size 5×5, whereas the degree of the resultant remains 4 in the coefficients of any polynomial. However, adding a monomial corresponding to the point $(1, 0)$ or $(0, 1)$ does not affect the the size of the Dixon matrix, since these points are Dixon-interior.

Consider the following generic polynomial system in which all terms of degree 2 are present: for $i = 0, 1, 2$:

$$c_{i,00} + c_{i,01}y + c_{i,02}y^2 + c_{i,10}x + c_{i,11}xy + c_{i,20}x^2$$

It can be viewed as a multihomogeneous of type $(1, 1; 2, 2)$, in which case it is multigraded, but then not generic, since its support does not contain all vertices that a system of type $(1, 1; 2, 2)$ can have, in particular monomial x^2y^2 is missing. But this system can also be viewed as multihomogeneous of type $(2; 2)$ in which case, all the monomial are present.

This example illustrates that it is not always possible to construct exact Dixon matrices. Dixon matrix is 5×5 and has extraneous factor of degree 1 in coefficients of any one of the equations. However, the Macaulay resultant formulation extracts the exact resultant in this by finding submatrix whose determinant is the extra factor; the resultant is expressed as a ratio of two determinants. There may not exist any resultant matrix whose determinant is precisely its resultant for this example, however.

5. BIVARIATE CASE

In this section, we discuss the case of $d = 2$, and give a precise characterization of when the Dixon resultant formulation leads to exact resultants and when extraneous factors are generated.

Let $\{\alpha, \beta, \gamma\}$ be a simplex in two dimensions. We will associate with it, the following two determinants:

$$|\alpha,\beta,\gamma| = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} x^{\alpha_x} y^{\alpha_y} & x^{\beta_x} y^{\beta_y} & x^{\gamma_x} y^{\gamma_y} \\ \overline{x}^{\alpha_x} y^{\alpha_y} & \overline{x}^{\beta_x} y^{\beta_y} & \overline{x}^{\gamma_x} y^{\gamma_y} \\ \overline{x}^{\alpha_x} \overline{y}^{\alpha_y} & \overline{x}^{\beta_x} \overline{y}^{\beta_y} & \overline{x}^{\gamma_x} \overline{y}^{\gamma_y} \end{vmatrix},$$

$$2 * \mathsf{Vol}_2(N(\{\alpha,\beta,\gamma\})) = \begin{vmatrix} 1 & \alpha_x & \alpha_y \\ 1 & \beta_x & \beta_y \\ 1 & \gamma_x & \gamma_y \end{vmatrix}.$$

In the expression for $|\alpha, \beta, \gamma|$, the total degree in $\overline{x}, x$ is $\alpha_x + \beta_x + \gamma_x - 1$, and it is the same for $\overline{y}, y$ as well. To simplify the analysis of the monomial set of $|\alpha, \beta, \gamma|$ and without any loss of generality, we substitute $\overline{x} = 1, \overline{y} = 1$.

5.1 Exact 2D Supports

DEFINITION 5.1. *A 2-dimensional simplex $\rho = \{\alpha, \beta, \gamma\}$ is orderable if β is strictly in between α, γ in every coordinate, i.e., ($\alpha_x < \beta_x \le \gamma_x$) and ($\alpha_y < \beta_y < \gamma_y$ or $\gamma_y < \beta_y < \alpha_y$).*

PROPOSITION 5.1. *A simplex $\rho = \{\alpha, \beta, \gamma\} \subseteq \mathcal{A}$ is Dixon-exact if and only if it is not orderable.*

PROOF. : W.l.o.g. assume that $\alpha_x \le \beta_x \le \gamma_x$. Consider the volume determinant expression of ρ:

$$2 * \mathsf{Vol}(\alpha,\beta,\gamma) = \underbrace{(\beta_x - \alpha_x)(\gamma_y - \beta_y)}_{V_1} - \underbrace{(\gamma_x - \beta_x)(\beta_y - \alpha_y)}_{V_2}.$$

The corresponding monomial determinant in the Dixon polynomial is

$$|\alpha,\beta,\gamma| = \underbrace{y^{\alpha_y} \frac{y^{\gamma_y} - y^{\beta_y}}{1-y} \frac{x^{\beta_x} - x^{\alpha_x}}{1-x}}_{P_1} - \underbrace{y^{\gamma_y} \frac{y^{\beta_y} - y^{\alpha_y}}{1-y} \frac{x^{\gamma_x} - x^{\beta_x}}{1-x}}_{P_2}.$$

Note $\mathsf{S_x}(P_1) \cap \mathsf{S_x}(P_2) = \oslash$, hence $|\mathsf{S_x}(|\alpha,\beta,\gamma|)| = |\mathsf{S_x}(P_1)| + |\mathsf{S_x}(P_2)|$. Since $V_1 = |\mathsf{S_x}(P_1)|$ and $V_2 = |\mathsf{S_x}(P_2)|$, then the only way that $V_1 - V_2 = |\mathsf{S_x}(|\alpha,\beta,\gamma|)|$ if $sign(V_1) = -sign(V_2)$; this implies that $sign(\gamma_y - \beta_y) = -sign(\beta_y - \alpha_y)$, i.e., β_y is not strictly in between α_y and γ_y. Since we have assumed that $\alpha_x \le \beta_x \le \gamma_x$ this amounts to ρ not being orderable. □

DEFINITION 5.2. *Given a support $\mathcal{A}$, define T to be a triangulation of $\mathcal{A}$, that is, for any distinct $\sigma, \rho \subseteq \mathcal{A}$ where $|\sigma| = |\rho| = 3$, $\sigma \in T$ and $\rho \in T$ if and only if $\mathsf{Vol}_2(N(\sigma) \cap N(\rho)) = 0$ and $\sum_{\sigma\in T} \mathsf{Vol}_2(N(\sigma)) = \mathsf{Vol}_2(N(\mathcal{A}))$.*

PROPOSITION 5.2. *Given a triangulation T of $\mathcal{A}$, for all distinct $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}, \rho = \{\rho_1, \rho_2, \rho_3\} \in T$,*

$$\mathsf{S_x}(|\sigma_1,\sigma_2,\sigma_3|) \cap \mathsf{S_x}(|\rho_1,\rho_2,\rho_3|) = \oslash$$

PROOF. : By checking all possible cases. □.

The above shows that if simplexes are geometrically disjoint, then the corresponding monomial sets in the Dixon polynomials are disjoint as well.

It is easy to see that a four point support $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$ has two triangulations.

PROPOSITION 5.3. *Let support $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$ and T_1 and T_2 two distinct triangulations of $\mathcal{A}$. If every simplex of $\mathcal{A}$ is Dixon-exact, then*

$$\mathsf{S_x}(\theta_{\mathcal{A}}) = \bigcup_{\sigma\in T_1} \mathsf{S_x}(|\sigma_1,\sigma_2,\sigma_3|) = \bigcup_{\sigma\in T_2} \mathsf{S_x}(|\sigma_1,\sigma_2,\sigma_3|)$$

PROOF. : Using Proposition 2.2,

$$\theta_{\mathcal{A}} = C_1|\alpha,\beta,\gamma| + C_2|\gamma,\beta,\delta| + C_3|\beta,\alpha,\delta| + C_4|\alpha,\gamma,\delta|$$

where $|\alpha, \beta, \gamma|$ is the monomial determinant corresponding to points α, β, γ, C_1 is the determinant of the corresponding coefficient matrix, and so on. For generic coefficients, $C_i \neq 0, 1 \le i \le 4$. W.l.o.g. assume that $\alpha_x \le \beta_x \le \gamma_x \le \delta_x$; then it can be shown that

$$|\alpha,\beta,\gamma| + |\gamma,\beta,\delta| = |\beta,\alpha,\delta| + |\alpha,\gamma,\delta|. \quad \square$$

PROPOSITION 5.4. *If every simplex of a support $\mathcal{A}$ without Dixon-interior points is Dixon-exact, then $\mathcal{A}$ is Dixon-exact.*

PROOF. : Proposition 2.2 implies that

$$\mathsf{S_x}(\theta_{\mathcal{A}}) = \bigcup_{\rho\subseteq\mathcal{A}} \mathsf{S_x}(|\rho_1,\rho_2,\rho_3|).$$

Given a triangulation T of $\mathcal{A}$, this can be rewritten as

$$\mathsf{S_x}(\theta_{\mathcal{A}}) = \bigcup_{T} \bigcup_{\rho\in T} \mathsf{S_x}(|\rho_1,\rho_2,\rho_3|).$$

It can be easily shown using Proposition 5.3 that any triangulation will yield the same set of monomials if all simplexes are Dixon-exact (see also [9]). That is, for any triangulation T of $\mathcal{A}$,

$$\mathsf{S_x}(\theta_{\mathcal{A}}) = \bigcup_{\rho\in T} \mathsf{S_x}(|\rho_1,\rho_2,\rho_3|).$$

The above implies that $|\mathsf{S_x}(\theta_{\mathcal{A}})| \le d!\mathsf{Vol}_2(N(\mathcal{A}))$, but $|\mathsf{S_x}(\theta_{\mathcal{A}})| \ge d!\mathsf{Vol}_2(N(\mathcal{A}))$ by Proposition 2.1, and hence, $|\mathsf{S_x}(\theta_{\mathcal{A}})| = d!\mathsf{Vol}_2(N(\mathcal{A}))$. □

PROPOSITION 5.5. *If $\mathcal{A}$ is Dixon-exact, then every simplex of $\mathcal{A}$ without Dixon-interior points is Dixon-exact.*

PROOF. : From the proof of the previous proposition,

$$\mathsf{S_x}(\theta_{\mathcal{A}}) \supseteq \bigcup_{\rho\in T} \mathsf{S_x}(|\rho_1,\rho_2,\rho_3|),$$

for any triangulation T of $\mathcal{A}$. In particular, if T contains a simplex which is not Dixon-exact, then using proposition 5.2,

$$|\mathsf{S_x}(\mathcal{A})| \ge \sum_{\rho\in T} |\mathsf{S_x}(|\rho_1,\rho_2,\rho_3|)| > d!\mathsf{Vol}_2(N(\mathcal{A})),$$

i.e., $\mathcal{A}$ is not Dixon exact if there exists a simplex in $\mathcal{A}$ that is not Dixon-exact.

THEOREM 5.1. *Given a generic unmixed polynomial system with support $\mathcal{A}$, $\mathcal{A}$ not including an orderable simplex is a necessary and sufficient condition for the Dixon formulation to compute its exact resultant.*

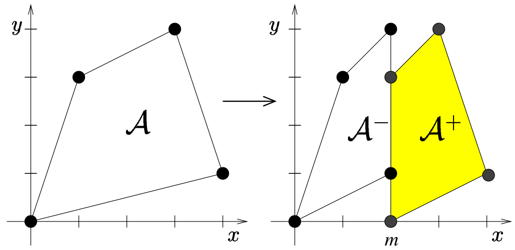

Figure 2: $\psi_{x=m}$ Operation on support $\mathcal{A}$

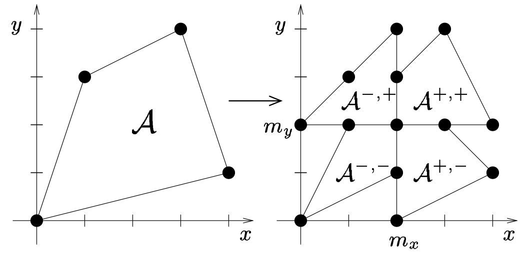

Figure 3: $\psi_{\substack{x=2\\y=2}}$ Operation on support $\mathcal{A}$

5.2 Estimating Degree of Extraneous Factor

Consider a support $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let m be a positive integer such that $\min_{i=1}^{n}(\alpha_i)_x < m < \max_{i=1}^{n}(\alpha_i)_x$. Consider the following two maps: $\psi^{\pm}_{x=m} : \mathbb{N}^d \to \mathbb{N}^d$ where for $a = (a_x, a_y) \in \mathbb{N}^2$,

$$\psi^{-}_{x=m}((a_x, a_y)) = \begin{cases} (m, a_y) & a_x > m, \\ (a_x, a_y) & a_x \leq m, \end{cases}$$

and also

$$\psi^{+}_{x=m}((a_x, a_y)) = \begin{cases} (m, a_y) & a_x < m, \\ (a_x, a_y) & a_x \geq m. \end{cases}$$

See Figure 2 for an example. These maps can be looked as maps on individual terms, where

$$\psi^{\pm}_{x=m}(\mathbf{x}^{\alpha}) = \mathbf{x}^{\psi^{\pm}_{x=m}(\alpha)},$$

or even on an entire polynomial, in which case ψ is applied to each term in the polynomial.

Let $\mathcal{A}^- = \psi^{-}_{x=m}(\mathcal{A})$ and $\mathcal{A}^+ = \psi^{+}_{x=m}(\mathcal{A})$. Note that $\mathcal{A}^- = S_{\mathbf{x}}(\psi^{-}_{x=m}(P_{\mathcal{A}}))$, where $P_{\mathcal{A}}$ is a generic polynomial whose support is $\mathcal{A}$. Abusing the notation, define for any $\alpha \in \mathcal{A}$, $\alpha^- = \psi^{-}_{x=m}(\alpha)$.

Note that $\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^-}) = \mathsf{S}_{\mathbf{x}}(\psi^{-}_{x=m}(\theta_{\mathcal{A}}))$ even though $\theta_{\mathcal{A}^-} \neq \psi^{-}_{x=m}(\theta_{\mathcal{A}})$. To see this, note that the above map $\psi^{\pm}_{x=m}$ will result in "splitting" various determinants in the Dixon polynomial (expressed using Proposition 2.2). It thus suffices to consider one of them. Below, we assume that $\alpha_x \leq \beta_x \leq m \leq \gamma_x$ and $\Delta = (1-x)(1-y)$.

$$\frac{1}{\Delta}\begin{vmatrix} x^{\alpha_x}y^{\alpha_y} & x^{\beta_x}y^{\beta_y} & x^{\gamma_x}y^{\gamma_y} \\ y^{\alpha_y} & y^{\beta_y} & y^{\gamma_y} \\ 1 & 1 & 1 \end{vmatrix} =$$

$$\frac{1}{\Delta}\begin{vmatrix} x^{\alpha_x}y^{\alpha_y} & x^{\beta_x}y^{\beta_y} & x^{m}y^{\gamma_y} \\ y^{\alpha_y} & y^{\beta_y} & y^{\gamma_y} \\ 1 & 1 & 1 \end{vmatrix} + \frac{1}{\Delta}\begin{vmatrix} x^{m}y^{\alpha_y} & x^{m}y^{\beta_y} & x^{\gamma_x}y^{\gamma_y} \\ y^{\alpha_y} & y^{\beta_y} & y^{\gamma_y} \\ 1 & 1 & 1 \end{vmatrix},$$

which splits into two disjoint monomial sets, since in the first determinant, $\deg(x) \leq m$, and all monomials of the second determinant have x^m as a factor. In the expression for the Dixon polynomial, these determinants are divided by $(1-x)(1-y)$, after which in the first determinant, the degree of $x < m$, whereas in the second determinant, the degree of $x \geq m$.

The following two identities hold:

- $\theta_{\mathcal{A}} = \psi^{-}_{x=m}(\theta_{\mathcal{A}}) + \psi^{+}_{x=m}(\theta_{\mathcal{A}})$, hence
$$\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}}) = \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^-}) \cup \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^+}).$$
- $\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^-}) \cap \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^+}) = \emptyset$.

We can also split the support on y. Define a map

$$\psi^{-}_{y=m}((a_x, a_y)) = \begin{cases} (a_x, m) & a_y > m, \\ (a_x, a_y) & a_y \leq m, \end{cases}$$

for $a = (a_x, a_y) \in \mathbb{N}^2$, and also

$$\psi^{+}_{y=m}((a_x, a_y)) = \begin{cases} (a_x, m) & a_y < m, \\ (a_x, a_y) & a_y \geq m. \end{cases}$$

Below assume that $\alpha_y \leq \beta_y \leq m \leq \gamma_y$.

$$\frac{1}{\Delta}|\alpha\beta\gamma| = \frac{1}{\Delta}y^{\gamma_y - m}|\alpha'\beta'\gamma| + \frac{1}{\Delta}y^{\alpha_y+\beta_y-2m}|\alpha\beta\gamma'|$$

where $\alpha' = (\alpha_x, m)$, $\beta' = (\beta_x, m)$ and $\gamma' = (\gamma_x, m)$. By a similar argument as for x, the two determinant in the sum have all disjoint monomials in y.

Maps $\psi^{\pm}_{y=m}$ can be viewed as partitioning maps for the support of the polynomial system, as well as the support of its Dixon polynomial in a same way as $\psi^{\pm}_{x=m}$. To make $\psi^{\pm}_{y=m}$ partitioning, we had to premultiply individual determinants in above expression.

Two maps $\psi^{\pm}_{x=m_x}$ and $\psi^{\pm}_{y=m_y}$ can be composed together, and the composition is commutative. By $\psi^{+,-}_{2,2}$ we will denote $\psi^{+}_{x=2} \circ \psi^{-}_{y=2}$. See Figure 3 where the polytope is first split on $x = 2$, and then each resulting polytope is split again on $y = 2$.

Proposition 5.6. *A point $p \in \mathcal{A}$ is Dixon-interior if there exists $a, b, c, d \in \mathcal{A}$ different from p such that*

$$\{a_y, d_y\} \leq p_y \leq \{b_y, c_y\} \quad \textit{and} \quad \{a_x, b_x\} \leq p_x \leq \{d_x, c_x\}.$$

Proof. : Since for all maps $\psi^{\pm}_{\substack{x=p_x\\y=p_y}}(q) = (p_x, p_y)$, for $q \in \{a, b, c, d\}$, the presence of p in $\mathcal{A}$ after partitioning is irrelevant, i.e. it does not influence the size of the support of the Dixon polynomial. □

Given a support $\mathcal{A}$ of a polynomial system P, we can apply ψ map on every point belonging to the Newton polytope of $\mathcal{A}$. After partitioning $\mathcal{A}$ into the smallest polytopes (i.e., when they cannot be partitioned any further due to lack of any integral points between any of the two coordinates), each resulting polytope can be shown to be Dixon-exact, and hence, the sum of the volumes of the smaller polytopes is the number of monomials in the Dixon polynomial.

As should be evident from maps ψ, Dixon-interior points do not contribute to the support of the Dixon polynomial,

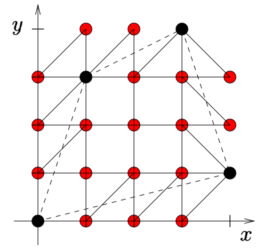

Figure 4: Complete partition of support $\mathcal{A}$

and hence the degree of the resultant. When ψ map is applied on the support as much as possible, Dixon-interior points in the support will be overlapped by other points in the support.

Given a 2-dimensional support, without computing resultant or even Dixon matrix for it, it is possible to exactly determine, in time proportional to the volume of the support, the size of the Dixon matrix and whether the projection operator computed using the Dixon formulation has an extraneous factor, and if it does, which support points cause the extraneous factor to appear.

Revisiting the example in Figure 2, after partitioning the support completely we get the set of supports shown in Figure 4. Note that each support in the partition is Dixon-exact. Their total volume is 19, and hence there will be 19 monomial in the Dixon polynomial. But $2 * \mathsf{Vol}_2(\mathcal{A}) = 18$, hence the Dixon matrix will yield an extra factor of degree 1 in the coefficients of each polynomial in the system.

After applying $\psi_{\substack{x=2\\y=2}}$ on the support, the total volume is still 18. It can be seen that the extra monomial appears in $\mathcal{A}^{-,+}$, which results from points $(0,0),(1,3),(3,4)$, which is exactly the case. Proposition 5.1 predicts this condition.

6. CONCLUSION

We have identified supports of unmixed generic polynomial systems for which the Dixon resultant formulation computes exact resultants. The main result is a generalization of the results reported in the literature about n-degree generic unmixed polynomial systems as well as about generic unmixed multigraded systems. For the bivariate case, an exact analysis is given whereby unmixed generic polynomial systems for which the Dixon resultant formulation computes exact resultants, are identified. A necessary and sufficient condition on the supports of such unmixed generic systems is given. Further, monomials in an unmixed generic polynomial system that contribute extraneous factors in resultant computations using the Dixon formulation are precisely characterized.

We are exploring ways to generalize the analysis for the bivariate case to the general d-dimensional case.

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