Exact Resultants for Corner-cut Unmixed Multivariate Polynomial Systems using the Dixon Formulation ⋆

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Abstract

Structural conditions on the support of a multivariate polynomial system are developed for which the Dixon-based resultant methods compute exact resultants. The concepts of a *corner-cut support* and *almost corner-cut support* of an unmixed polynomial system are introduced. For generic unmixed polynomial systems with corner-cut and almost corner-cut supports, the Dixon based methods can be used to compute their resultants exactly. These structural conditions on supports are based on analyzing how such supports differ from box supports of d-degree systems for which the Dixon formulation is known to compute resultants exactly. Such an analysis also gives a sharper bound on the complexity of resultant computation using the Dixon formulation in terms of the support and the mixed volume of the Newton polytope of the support.

These results are a direct generalization of the authors' results on bivariate systems including the results of Zhang and Goldman as well as of Chionh for generic unmixed bivariate polynomial systems with corner-cut supports.

Key words: Resultant, Dixon Method, Extraneous Factor, BKK Bound, Support, Support hull, Corner-Cut, n-degree Systems.

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1 Introduction

Resultant matrices based on the Dixon formulation have turned out to be quite efficient in practice for simultaneously eliminating many variables on a variety of examples from different application domains; for details and comparison with other resultant formulations and elimination methods, see [1,2] and `http://www.cs.unm.edu/∼artas`. Necessary conditions can be derived on parameters in a problem formulation under which the associated polynomial system has a solution.

A main limitation of matrix-based approaches for computing resultants is that often an extraneous factor is generated [3] with no relation to the resultant of a given polynomial system. This paper reports results about polynomial systems for which the Dixon formulation leads to the exact resultant (without any extraneous factor). The concepts of a *corner-cut support* and *almost corner-cut support* of unmixed polynomial systems are introduced; these notions are based on analyzing how such supports deviate from the support of the associated d-degree system whose resultant can be computed exactly using the Dixon formulation. It is proved that for generic unmixed polynomial systems with corner-cut supports and almost corner-cut supports, the Dixon based resultant methods can compute their resultants exactly. These results generalize the earlier results of the authors for bivariate polynomial systems [2] as well as the results of [4] and [5] on corner-cut supports for bivariate polynomial systems.

This approach has the distinct advantage of generalizing the most known cases of unmixed polynomial systems (such as d-degree systems as well as systems with corner-cut supports) for which the Dixon formulation is known to compute the resultant exactly [6,2].

The paper bring together the results of [7], [8] and builds on the results proved in [7] in which it is shown that the degree of the projection operator computed using Dixon resultant formulations is determined solely by the support hull of the support of a polynomial system. This relationship further generalizes the results in [8] and provides insight into the construction of Dixon matrices. Approximations to an upper bound on the size of the Dixon matrix and the degree of a projections operator are compared showing that a detailed analysis of the projections of the support hull yields a tighter upper bound. The paper also elaborates on many of the results in [8] providing with detailed proofs and demonstrates how these results strictly generalize our earlier results about the bivariate case.

The focus in this paper is on the use of the generalized Dixon resultant formulation for computing resultants and projection operators as introduced in

[9]. The results also apply to the Dixon multiplier matrices introduced [10]. It is proved in [10] that for a generic unmixed polynomial system, if the Dixon formulation produces a Dixon matrix whose determinant is the resultant, then the determinant of the corresponding Dixon multiplier matrix based on the construction in [10] is also the resultant. In case the Dixon matrix is such that the determinant of its maximal minor has an extraneous factor besides the resultant, the maximal minor of the corresponding Dixon multiplier matrix does not have an extraneous factor of higher degree. (A Dixon multiplier matrix is a Sylvester-type resultant sparse matrix in which entries are either zeros or coefficients of terms in polynomials in the polynomial system, and it is constructed using the Dixon formulation; for more details, see [10].)

1.1 Overview

The next section discusses preliminaries and background – the concept of a multivariate resultant of a polynomial system, the support of a polynomial and the degree of the resultant as determined by the BKK bound based on the mixed volume of the Newton polytopes of the supports of the polynomials in a polynomial system.

Section 3 is a review of the generalized Dixon formulation including the Dixon polynomial and Dixon matrix. The section concludes with a discussion of how the Cauchy-Binet expansion of determinants can be used to show that the Dixon polynomial and its support are related to the support of the polynomials in the polynomial system. The size of the Dixon matrix of a polynomial system is determined by the size of the support of the associated Dixon polynomial. It is shown how the support of the Dixon polynomial is affected when the support of the given polynomial system is translated. The rest of the paper is about generic unmixed polynomial systems whose support is cornered, i.e., situated at the origin (meaning that every polynomial includes a constant term).

Section 4 discusses the concept of a support hull and its interior. This concept turns out to be more useful for relating the size of the Dixon matrix of a given polynomial system to its support. It has been established in [7] that the generic inclusion of a term whose exponent is support hull interior of the support of a given generic unmixed polynomial system does not change the size of the Dixon matrix of the modified polynomial system.

Section 5 reviews the results about generic unmixed bivariate polynomial systems. The concept of a corner-cut support is discussed; for generic unmixed polynomial systems, the support-hull of their support being corner-cut is both a necessary and sufficient condition for the Dixon-based resultant methods to compute resultants without any extraneous factors. More details can be found in [2].

Section 6 generalizes the concepts and notations introduced to study bivariate polynomial systems to arbitrary dimension. By a combinatorial analysis of the deviation of a given support from that of an d-degree polynomial system, conditions are identified on a support for which the generalized Dixon formulation computes exact resultants (up to a sign). By considering projections of the support complement of the support of a given polynomial system with respect to the associated d-degree systems (whose support is the bounded box enclosing the support of the polynomial system) using a given variable order, a formula is derived for the size of the Dixon matrix in terms of the size of the Dixon matrix for the associated d-degree system and the size of various projections of the support complement.

Section 7 discusses conditions on the support of a polynomial system and their projections which lead to a Dixon matrix with the appropriate size such that its determinant is the resultant. The concept of a d-dimensional corner-cut support is introduced which generalizes the notion of corner-cut support for the bivariate case discussed in [2,4,5]. It is shown that for a generic unmixed polynomial system with a corner-cut support, the Dixon formulation computes the resultant exactly. The requirement in the structural condition defining a corner-cut support can be relaxed while still preserving the property of the associated Dixon matrix that its determinant is the resultant. The concept of an *almost* corner-cut support (which has no analog in the bivariate case) is introduced. For a generic unmixed polynomial system with an almost corner-cut support, the Dixon method also produces a matrix whose determinant is resultant. The notion of support-interior point within a support is generalized from the bivariate case to d variables; it is shown that a given unmixed polynomial system can be modified to generically include the term corresponding to a support-interior point in its support without affecting the size of the Dixon matrix.

2 Support of a Polynomial System and Degree of Resultant

A resultant is a necessary and sufficient condition for the existence of solutions on the coefficients of a polynomial system in a variety. Here we consider a toric resultant, which is the condition for the existence of a solution with non-zero coordinates (more information can be found in [11–13]). The resultant defined for an over-constrained polynomial system $\mathcal{F} = \{f_0, f_1, \ldots, f_d\}$ with

$$f_0 = \sum_{\alpha \in \mathcal{A}_0} c_{0,\alpha} \mathbf{x}^{\alpha}, \quad f_1 = \sum_{\alpha \in \mathcal{A}_1} c_{1,\alpha} \mathbf{x}^{\alpha}, \qquad \cdots, \qquad f_d = \sum_{\alpha \in \mathcal{A}_d} c_{d,\alpha} \mathbf{x}^{\alpha},$$

where $\mathcal{A}_i \subset \mathbb{N}^d$, $\alpha = (\alpha_1, \cdots, \alpha_d)$, and a monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$. In general, the structure of the resultant is dependent on the set of monomials appearing in the polynomial system. $\mathcal{A}_i$ is called the *support* of a polynomial $f_i \in \mathbb{Q}[\mathbf{c}][x_1, \ldots, x_d]$.

The (lattice) convex hull of the support of a polynomial f is called its Newton polytope. One can relate the Newton polytopes of a polynomial system to the number of its roots, but first, we define a special function called the *mixed volume*, on supports.

Definition 1 ([11,12]) *The* ***mixed volume function*** $\mu(\mathcal{Q}_1, \ldots, \mathcal{Q}_d)$, *where* $\mathcal{Q}_i$ *is a convex hull, is a unique function which is multi-linear with respect to Minkowski sum and scaling, and is defined to have the multi-linear property:*

$$\mu(\mathcal{Q}_1, \ldots, a\mathcal{Q}_k + b\mathcal{Q}'_k, \ldots, \mathcal{Q}_d) = a\,\mu(\mathcal{Q}_1, \ldots, \mathcal{Q}_k, \ldots, \mathcal{Q}_d) + b\,\mu(\mathcal{Q}_1, \ldots, \mathcal{Q}'_k, \ldots, \mathcal{Q}_d);$$

To ensure uniqueness, $\mu(\mathcal{Q}, \ldots, \mathcal{Q}) = d!\mathsf{Vol}(\mathcal{Q})$, where $\mathsf{Vol}()$ is the Euclidean volume of the convex hull of $\mathcal{Q}$.

In this paper, toric resultants are considered. The set $\mathbb{C}^* = \mathbb{C} - 0$, i.e., the set of complex numbers without zero, is referred as algebraic torus. Varieties which include the d-dimensional algebraic torus $(\mathbb{C}^*)^d$ are called toric; the condition for the existence of a solution in a toric variety is called a toric resultant. The number of toric solutions of a given polynomial system and the degree of its toric resultant are governed by the mixed volume as stated by the following theorem.

Theorem 2 (BKK Bound) *Given a polynomial system* $\mathcal{F} = \{f_1, \ldots, f_d\}$ *in* d *variables* $\{x_1, \ldots, x_d\}$ *with the support* $\langle \mathcal{A}_0, \ldots, \mathcal{A}_d \rangle$, *the number of roots in* $(\mathbb{C}^*)^d$, *counting multiplicities, of the polynomial system is either infinite or*

$$\#\mathsf{Roots}(f_1, \ldots, f_d) \leq \mu\left(\mathcal{A}_1, \ldots, \mathcal{A}_d\right);$$

the inequality becomes equality when the coefficient of polynomials in the system satisfy the genericity requirements.

Since we are interested in over-constrained polynomial systems, usually consisting of $d+1$ polynomials in d variables, the BKK bound tells us the degree of the resultant.

In the resultant, the degree of the coefficients of f_0 is equal to the number of common roots of the rest of the polynomials in $\mathcal{F}$. It is possible to choose any other polynomial f_i instead of f_0. The resultant expression can also be obtained by substituting into f_i, the common roots of the remaining polynomials in $\mathcal{F}$. Thus, the degree of the coefficients of f_i in the resultant equals the number of roots of the remaining set of polynomials in $\mathcal{F}$.

Definition 3 *A polynomial system* $\mathcal{F} = \{f_0, f_1, \ldots, f_d\}$ *with the corresponding supports* $\mathcal{A}_0, \ldots, \mathcal{A}_d$ *is called* ***unmixed*** *if* $\mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_d$*; otherwise, if* $\mathcal{A}_i \neq \mathcal{A}_j forsomei, j$*, then* $\mathcal{F}$ *is called* ***mixed****.*

This paper considers unmixed polynomial systems in which all polynomials have the same structure. Therefore, for notational convenience, we will drop the index of a support $\mathcal{A}_i$ and say that the support of the polynomial system $\mathcal{F}$ is $\mathcal{A}$, if the support of every polynomial in it is $\mathcal{A}$ (i.e., $\mathcal{A}_i = \mathcal{A}_j = \mathcal{A}$).

For an unmixed polynomial system $\mathcal{F}$ with a support $\mathcal{A}$, there is an easy formula for the degree of the resultant:

$$\deg_{f_i} \mathrm{Res} = d!\mathsf{Vol}(\mathcal{A}),$$

where $\deg_{f_i} \mathrm{Res}$ is the degree of the toric resultant in the coefficients of every polynomial f_i. Knowing the degree of the resultant a priori is useful for identifying cases for which a given method for computing the resultant is exact (i.e., the method does not produce a result with any extraneous information).

In the next section, we first give a brief overview of the Dixon formulation by defining the concepts of the Dixon polynomial and the Dixon matrix of a given polynomial system. Expressing the Dixon polynomial using the Cauchy-Binet expansion of determinants of a matrix is useful for illustrating the dependence of the construction on the support of a given polynomial system.

3 Dixon Matrix

In [14], Dixon generalized the Bezout-Cayley construction for computing the resultant of two univariate polynomials to the bivariate case. In [9], Kapur, Saxena and Yang further generalized this construction to the general multivariate case; the concepts of a Dixon polynomial and a Dixon matrix were introduced in [9] as well. Below, the generalized multivariate Dixon formulation for simultaneously eliminating many variables from a polynomial system and computing its resultant is reviewed. More details can be found in [1].

In contrast to multiplier matrices such as a Sylvester matrix, a Macaulay matrix and a sparse resultant matrix a la Sturmfels et al [15,16], a Dixon matrix is dense since its entries are determinants of the coefficients of the polynomials in the original polynomial system. It has the advantage of being an order of magnitude smaller in comparison to a multiplier matrix, which makes the method efficient since the computation of the determinant of a matrix with symbolic entries is sensitive to its size. The Dixon matrix is constructed through the computation of the Dixon polynomial, which can be expressed in a matrix form.

Let $\pi_i(\mathbf{x}^\alpha) = \overline{x}_1^{\alpha_1} \cdots \overline{x}_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} \cdots x_d^{\alpha_d}$, where $i \in \{0, \ldots, d\}$, and $\overline{x}_i$'s are new variables; thus, $\pi_0(\mathbf{x}^\alpha) = \mathbf{x}^\alpha$. The function π_i is extended to polynomials in a natural way as:

$$\pi_i(f(x_1, \ldots, x_d)) = f(\overline{x}_1, \ldots, \overline{x}_i, x_{i+1}, \ldots, x_d),$$

obtained by substituting $\overline{x}_j$ for x_j in $f, 1 \leq j \leq i$.

Definition 4 *Given a polynomial system $\mathcal{F} = \{f_0, f_1, \ldots, f_d\}$, where $\mathbf{c} = \{c_{i,\alpha} | \alpha \in \mathcal{A}\}$ and $\mathcal{F} \subset \mathbb{Q}[\mathbf{c}][x_1, \ldots, x_d]$, its **Dixon polynomial** is*

$$\theta(f_0, \ldots, f_d) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} \pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\ \pi_1(f_0) & \pi_1(f_1) & \cdots & \pi_1(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(f_0) & \pi_d(f_1) & \cdots & \pi_d(f_d) \end{vmatrix}. \quad (1)$$

Hence, $\theta(f_0, f_1, \ldots, f_d) \in \mathbb{Q}[\mathbf{c}][x_1, \ldots, x_d, \overline{x}_1, \ldots, \overline{x}_d]$, where $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_d$ are new variables.

The order in which the original variables in $\mathbf{x}$ are replaced by new variables in $\overline{\mathbf{x}}$ is significant in the sense that the Dixon polynomial computed using two different orderings can be different.

Definition 5 *The Dixon polynomial $\theta(f_0, \ldots, f_d)$ can be written in* bilinear form *as*

$$\theta(f_0, f_1, \ldots, f_d) = \overline{X} \Theta X^T,$$

*where $\overline{X} = [\overline{\mathbf{x}}^{b_1}, \ldots, \overline{\mathbf{x}}^{b_k}]$ and $X = [\mathbf{x}^{a_1}, \ldots, \mathbf{x}^{a_l}]$ are row vectors. The $k \times l$ matrix Θ is called the **Dixon matrix**.*

It has been shown in [17] that Θ is a resultant matrix. The resultant of $\mathcal{F}$ can thus be extracted from a *projection operator*, which is the determinant of some maximal minor of Θ.

Each entry in Θ is a polynomial in the coefficients of the original polynomials in $\mathcal{F}$; moreover, its degree in the coefficients of any given polynomial is at most 1. Therefore, a projection operator computed using the Dixon formulation can be of at most of degree $|X|$ in the coefficients of any single polynomial $f_i \in \mathcal{F}$.

We are interested in identifying conditions when the resultant matrix (or the Dixon matrix) Θ is **exact**, i.e., its determinant is exactly (up to a constant factor) the resultant. Also, when Θ is not exact, we are interested in predicting an extraneous factor in a projection operator computed from Θ (at the very least, the degree of the extraneous factor). In the unmixed case,

$$|X| \geq d! \ \mathsf{Vol}(\mathcal{A}).$$

We are thus interested in analyzing the size and structure of the monomial set X. The size of X tells the number of columns in Θ. In the generic case, if $|X| = d! \ \mathsf{Vol}(\mathcal{A})$, then Θ is exact; otherwise, it is not exact.

We will relate the support $\mathcal{A}$ of a given unmixed polynomial system $\mathcal{F}$ to the support of its Dixon polynomial X and to the size of the associated Dixon matrix.

3.1 Relating Size of Dixon Matrix to Support of a Polynomial System

There is a different formula for the Dixon polynomial based on the Cauchy-Binet expansion of the determinant of the product of two non-square matrices. Given a generic unmixed polynomial system $\mathcal{F}$ with a support $\mathcal{A}$, denote by $\sigma \Subset \mathcal{A}$, a **simplex** $\sigma = \langle \sigma_0, \sigma_1, \ldots, \sigma_d \rangle$, where $\sigma_i \in \mathcal{A}$.

Proposition 6 (Cauchy-Binet Expansion) *Given an unmixed polynomial system $\mathcal{F} = \{f_0, f_1, \ldots, f_d\}$ with a support $\mathcal{A}$,*

$$\theta(f_0, f_1, \ldots, f_d) = \sum_{\substack{\sigma \Subset \mathcal{A} \\ |\sigma| = d+1}} \sigma(\mathbf{c}) \ \sigma(\mathbf{x}) = \sum_{\substack{\sigma \Subset \mathcal{A} \\ |\sigma| = d+1}} \theta_\sigma,$$

where $\theta_\sigma = \sigma(\mathbf{c}) \ \sigma(\mathbf{x})$ and

$$\sigma(\mathbf{c}) = \begin{vmatrix} c_{0,\sigma_0} & c_{0,\sigma_1} & \cdots & c_{0,\sigma_d} \\ c_{1,\sigma_0} & c_{1,\sigma_1} & \cdots & c_{1,\sigma_d} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d,\sigma_0} & c_{d,\sigma_1} & \cdots & c_{d,\sigma_d} \end{vmatrix},$$

$$\sigma(\mathbf{x}) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} \pi_0(\mathbf{x}^{\sigma_0}) & \pi_0(\mathbf{x}^{\sigma_1}) & \cdots & \pi_0(\mathbf{x}^{\sigma_d}) \\ \pi_1(\mathbf{x}^{\sigma_0}) & \pi_1(\mathbf{x}^{\sigma_1}) & \cdots & \pi_1(\mathbf{x}^{\sigma_d}) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(\mathbf{x}^{\sigma_0}) & \pi_d(\mathbf{x}^{\sigma_1}) & \cdots & \pi_d(\mathbf{x}^{\sigma_d}) \end{vmatrix}.$$

PROOF. The proof follows from the multi-linearity property of determinants; see [10] for details.

The above identity shows that if generic coefficients are assumed in $\mathcal{F}$, then the support of the Dixon polynomial depends entirely on the support of $\mathcal{F}$ as $\sigma(\mathbf{c})$'s do not vanish or cancel each other. To emphasize the dependence of θ on the support $\mathcal{A}$ of $\mathcal{F}$, the above identity is also written as $\theta_{\mathcal{A}} = \sum_{\sigma \in \mathcal{A}} \theta_\sigma$.

The **support** of the Dixon polynomial,

$$\Delta_{\mathcal{A}} = \{ \alpha \mid \mathbf{x}^\alpha \in \theta(f_0, \ldots, f_d) \}.$$ [1]

Further,

$$\Delta_{\mathcal{A}} = \bigcup_{\sigma \in \mathcal{A}} \Delta_\sigma, \qquad \text{where} \quad \Delta_\sigma = \{ \alpha \mid \mathbf{x}^\alpha \in \theta_\sigma \}.$$

The following proposition shows that the translation of the support of polynomials in an unmixed system has no effect on the size of the support of the Dixon polynomial (and hence, the size of the Dixon matrix).

Proposition 7 *Given an unmixed polynomial system $\mathcal{F}$ with a support $\mathcal{A}$, let $q = (q_1, \ldots, q_d)$, where $q_i = \min_{\alpha \in \mathcal{A}} \alpha_i$, then*

$$\Delta_{\mathcal{A}} = \{(q_1, 2\,q_2, \ldots, dq_d)\} + \Delta_{\mathcal{A}-\{q\}}\,^{2},$$

that is, $\Delta_{\mathcal{A}}$ is the appropriate "shift" of the support of the Dixon polynomial of the corresponding polynomial system whose support is situated at the origin.

PROOF. Since $\mathcal{A}$ is the support of polynomials $\{f_0, f_1, \ldots, f_d\}$,

$$f_0 = \mathbf{x}^q g_0, \quad f_1 = \mathbf{x}^q g_1, \quad \cdots, \quad f_2 = \mathbf{x}^q g_d,$$

where $\mathcal{A} - \{q\}$ is the support of $\{g_0, g_1, \ldots, g_d\}$. Therefore

$$\theta(f_0, f_1, \ldots, f_d) = x_1 x_2^{2\,q_2} \cdots x_d^{dq_d}\, \overline{x}_1^{d\,q_1}\, \overline{x}_1^{(d-1)q_2} \cdots \overline{x}_d\, \theta(g_0, g_1, \ldots, g_d),$$

by factoring monomials from the rows of the matrix in the expression for the Dixon polynomial as given in (1). □

Henceforth, it will be assumed, without any loss of generality, that in the unmixed case, $\mathcal{A}$ is situated at the origin, that is, $\min_{\alpha \in \mathcal{A}} \alpha_i = 0$ for $i = 1, \ldots, d$

In the subsequent section, the key results of [5],[4] and [2], which characterize supports of generic unmixed bivariate polynomial systems for which the Dixon-based resultant matrices (both Dixon matrices and Dixon multiplier matrices) are exact, are generalized.

[1] By an abuse of notation, by $\mathbf{x}^\alpha \in f$ we mean that $\mathbf{x}^\alpha$ appears in (the simplified form of) the polynomial f with a non-zero coefficient, i.e., α is in the support of f.
[2] "$-$" is the regular vector subtraction.

4 Support Hull

The concept of a support hull of a given support $\mathcal{A}$ is introduced. This concept has to be shown in [7] to be critical in determining the size of the Dixon matrix associated with a given polynomial system $\mathcal{F}$, much like the convex hull of the support determines the degree of the toric resultant of $\mathcal{F}$.

Given two points on a line one can say that one point is before the other in some direction. Going to higher dimensions, such relationship between the points can be extended rectilinearly as follows.

Definition 8 *Given $k \in \mathbb{Z}_2^d$ and points $p, q \in \mathbb{N}^d$,*

$$p \underset{k}{<} q \quad \text{if} \quad \begin{cases} p_j < q_j \ \text{if}\ k_j = 1, \\ p_j \geq q_j \ \text{if}\ k_j = 0. \end{cases}$$

Any $k \in \mathbb{Z}_2^d$ is called an octant; if $d = 2$, it is called a quadrant. Note that from above, p_j is strictly smaller that q_j; from below, p_j is equal or greater that q_j. When equality is also allowed from above, the relation is denoted by $p \underset{k}{\leq} q$. For a fixed k, this relation is transitive, but it is not a total order.

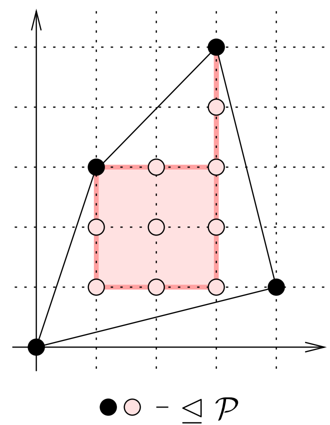

Figure 1: Support Hull

Definition 9 *Given a support $\mathcal{A} \subset \mathbb{N}^d$,*

$$\texttt{SupportHull}(\mathcal{A}) = \{\, p \mid \forall\, k \in \mathbb{Z}_2^d,\ \exists\, q \in \mathcal{A} \text{ such that } p \underset{k}{\leq} q \,\}.$$

Also, $p \trianglelefteq \mathcal{A}$ if $p \in$ $\texttt{SupportHull}(\mathcal{A})$.

The support hull of a given support is thus a minimal object rectilinearly connecting all points in a support. In contrast to the convex hull of a support, the support hull is not a connected set. Figure 1 shows a support (filled points) and its support hull (all points).

Definition 10 *A point $p \in \mathbb{N}^d$ is called **support hull interior** w.r.t. a support $\mathcal{A}$ if for all octants $k \in \mathbb{Z}_2^d$, there exists $q \in \mathcal{A}$, where $p \neq q$, such that $p \underset{k}{\leq} q$, i.e. every octant of p contains point from $\mathcal{A}$.*

From the definition, it can be easily be seen that a support hull interior point wrt to the support hull of a given support is in the convex hull of the support.

Theorem 11 *[7] Given two generic unmixed polynomial systems with cor-*

nered supports $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\texttt{SupportHull}(\mathcal{P}) = \texttt{SupportHull}(\mathcal{Q}), \quad \textit{then} \quad \Delta_{\mathcal{P}} = \Delta_{\mathcal{Q}},$$

i.e., the Dixon matrices generated from the two polynomial systems are of the same size.

The following corollary is an immediate consequence of the above theorem.

Corollary 12 *The size of the Dixon matrix of a generic unmixed polynomial system $\mathcal{F}$ is invariant under the generic inclusion of a monomial $\mathbf{x}^{\alpha}$ into $\mathcal{F}$ whose exponent α is support interior w.r.t. the support of $\mathcal{F}$.*

Such dependence of the construction and the size of the Dixon matrix on the support of a polynomial system suggests a way to identify polynomial systems for which the Dixon construction will result in projection operators with extraneous factors.

If the support of a generic unmixed polynomial system contains a point which is in the interior of the convex hull of the support, but is not support hull interior, the Dixon construction can result in a projection operator with an extraneous factor. Unfortunately, the converse does not hold since there are examples of generic unmixed systems with no such points for which the Dixon construction produces projection operators with extraneous factors. In subsequent sections, we will give a more precise description of supports which are exact under the Dixon construction. We first review known results for the bivariate case. We then generalize the concepts and results to the general d variable case.

5 Bivariate case: Corner-Cut Systems

In [5], Zhang and Goldman identified a necessary condition on the support of a generic unmixed bivariate polynomial system for which an exact Sylvester-like resultant matrix can be constructed. In [4], Chionh showed that under this condition, the Dixon matrix is also exact. In [2], we have generalized these results by establishing that this condition on supports is not only necessary but also sufficient for both exact Sylvester-like matrices as well as exact Dixon matrices. The concept of a support-interior point in a support generalizes these notions even further. First, we review these concepts and results for the bivariate case; later, we show how they can be generalized for an arbitrary number of variables.

Given a cornered support $\mathcal{A}$, such that $\mathcal{A} = \{a_1, \dots, a_n\}$, let $b_j = \max_{i=1}^{n} a_{i,j}$, for $j = 1, \dots, d$.

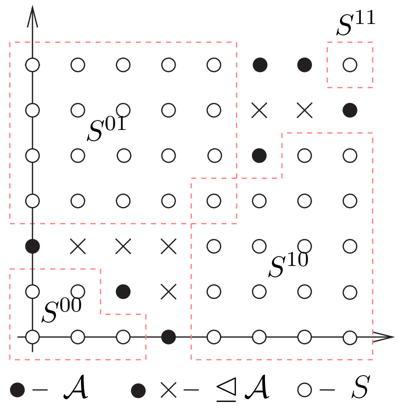

Fig. 2. Support Hull Complement

Definition 13 *Given a cornered support $\mathcal{A}$, the **box support** of $\mathcal{A}$ is:*

$$\mathcal{B} = \{\ p = (p_1, \ldots, p_d) \mid 0 \le p_j \le b_j\ \}.$$

A generic unmixed polynomial system $\mathcal{F}$ with a box support $\mathcal{B}$ is called in [17,18] d-degree system. It has been proved in [17,18] that for a d-degree system, the Dixon matrix is exact.

Definition 14 *Given a generic unmixed bivariate polynomial system with a cornered support $\mathcal{A}$, let for $k \in \mathbb{Z}_2^2$,*

$$S^k = \{\, s \mid s \in \mathcal{B} \ \ \textit{and for all}\ \alpha \in \mathcal{A},\ s \not\preceq_k \alpha \,\} \quad \textit{and} \quad S = \bigcup_{k \in \mathbb{Z}_2^2} S^k.$$

As a consequence of Definitions 9 and 14, it is easy to see that

$$S = \mathcal{B} - \texttt{SupportHull}(\mathcal{A}).$$

See Figure 2, where the hollow points belong to S and the crossed points are in the support hull interior of $\mathcal{A}$.

Definition 15 *A bivariate polynomial system support $\mathcal{A}$ is called **corner-cut** if for each $k \in \mathbb{Z}_2^2$, the corresponding set S^k is* rectangular.

Theorem 16 *[2] A generic bivariate unmixed polynomial system is Dixon exact if and only if its support is corner-cut.*

In the later sections, we generalize the above theorem for the general multivariate case. It will be clear that a straight forward generalization, in which S^k are multidimensional rectangles, is not appropriate. For example, see Figure 4 where it is possible to describe the support as rectangular regions removed from corners, yet the Dixon matrix generated from an generic unmixed system with this support is not exact. The main reason for this is the crucial role of variable order played in the Dixon construction; this role is not evident in the bivariate case.

6 Multivariate Case: Upper bound on Size of the Dixon Matrix

The complexity of the resultant computation using the Dixon formulation for a given polynomial system is governed by the size of the generated Dixon matrices. An upper bound on the size of the Dixon matrix for a generic unmixed polynomial system was proved in [17] to be the Minkowski sum of successive projections of the supports of the polynomials in the polynomial system. Depending on the variable order used in the construction (Definition 4), different Dixon polynomials and different Dixon matrices can be obtained, and more importantly, the size of the resulting Dixon matrices might not be the same. Dixon polynomials of smaller size will result in smaller Dixon (as well as Dixon multiplier) matrices; therefore, extraneous factors in the projection operators are relatively of smaller degrees.

In this section, a tighter bound on the size of the Dixon matrix is established by analyzing how the support of a given generic unmixed polynomial system differs from the support of the associated d-degree polynomial system.

6.1 d-Degree System and its Box Support

It has been proven in [17,18] that the Dixon matrix is exact for a generic polynomial system with a d-degree support.

Proposition 17 *The support of the Dixon polynomial of a d-degree system with a box support $\mathcal{B}$,*

$$\Delta_{\mathcal{B}} = \{\ p = (p_1, \ldots, p_d) \mid 0 \le p_i < i\ b_i\ \} \quad \textit{and} \quad |\Delta_{\mathcal{B}}| = d!\ \mathsf{Vol}(\mathcal{B}) = d! \prod_{i=1}^{d} b_i.$$

As in the case of bivariate systems,we relate the difference between the supports $\Delta_{\mathcal{B}}$ and $\Delta_{\mathcal{A}}$ to the difference between the box support $\mathcal{B}$ and the support $\mathcal{A}$. However, unlike the bivariate case, this relation is analyzed by investigating various projections of $\mathcal{A}$ along different coordinates. Unlike the bivariate case, it is not sufficient to analyze the projection using a single variable order. We will then derive tighter upper bounds on the size of $\Delta_{\mathcal{A}}$ and the size of the associated Dixon matrix.

6.2 Bounds derived in [17]

It was proven in [17] that $\Delta_{\mathcal{A}}$ is contained in the convex hull of the Minkowski sum of the successive projections of $\mathcal{A}$.

Theorem 18 *[17] Given a generic unmixed polynomial system $\mathcal{F}$ with a support $\mathcal{A}$,*

$$\Delta_{\mathcal{A}} \subseteq \texttt{ConvexHull}\left(\sum_{i=0}^{d} \pi_i(\mathcal{A})\right) \ \cap \ \mathbb{Z}^d,$$

where $\pi_i(\mathcal{A}) = \{\, \pi_i(\alpha) \,|\, \alpha \in \mathcal{A}\,\}$ and $\pi_i(\alpha) = (0, \ldots, 0, \alpha_{i+1}, \ldots, \alpha_d)$ for $i \in \{0, \ldots, d\}$.

Earlier, π_i is defined on polynomials and monomials, where for $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\pi_i(\mathbf{x}^\alpha) = \overline{x}_1^{\alpha_1} \cdots \overline{x}_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} \cdots x_d^{\alpha_d}$. Since the support of a single monomial $\pi_i(\mathbf{x}^\alpha)$ is $(0, \ldots, 0, \alpha_{i+1}, \ldots, \alpha_d)$, the use of projection π_i on a support as in the statement of the above theorem is consistent with the earlier definition.

One can easily see that in the expression for the Dixon polynomial (Definition 4), the support of the determinant of the matrix is contained in the Minkowski sum of the successive projection of $\mathcal{A}$; division by $\overline{x}_i - x_i$ does not introduce any new points outside the convex hull of this sum.

6.3 Tighter Bounds using Support Hull

In the statement of the above theorem, the convex hull can be replaced by the support hull and the proof still goes through. This is due to the property that the construction of the Dixon matrix depends on the support hull of a support of a polynomial system (see [7] for details). Hence,

$$|\Delta_{\mathcal{A}}| \leq \left|\mathsf{SupportHull}\left(\sum_{i=0}^{d} \pi_i(\mathcal{A})\right)\right|.$$

Since the the support hull of a support is contained in its convex hull, a bound based on the support hull is almost always better than the corresponding bound based on the convex hull. Figure 3 is an example for which an upper bound based on the convex hull of a support is worse than its support hull. (The shaded area in the figure corresponds to the support hull). The number of lattice points in the sum of projections (the outline of the figure) is 39, where as the number of points in the support hull of the sum of the projections (in the shaded area in the figure) is 38. (For this example, the size of the Dixon polynomial $|\Delta_{\mathcal{A}}| = 23$).

Consider the case where a 2D support contains only 3 points $\{(0,0),(9,0),(0,9)\}$. The number of points in the sum of the projections of the support is 145, whereas the number of points in the support hull of projections is 109.

The difference between the support hull bound and the actual size of the Dixon matrix, i.e. $|\Delta_{\mathcal{A}}|$, suggests a need for further analysis to develop a tighter upper bound. Particularly, $\Delta_{\mathcal{A}}$ does not include "upper" boundary points of the support hull (hollow, non-shaded boundary points in Figure 3). Because of the division done to compute the Dixon polynomial, the degrees of monomials are reduced. When this is considered, an exact bound for the bivariate case can be obtained; see [2] for a complete analysis.

For the general multivariate case, it is proved in [7] that the size of Dixon matrix for an unmixed generic polynomial system with a support $\mathcal{A}$ is bounded by the number of points in the support hull of the sum of projections minus the "upper" boundary points of the support hull; details can be found in [7].

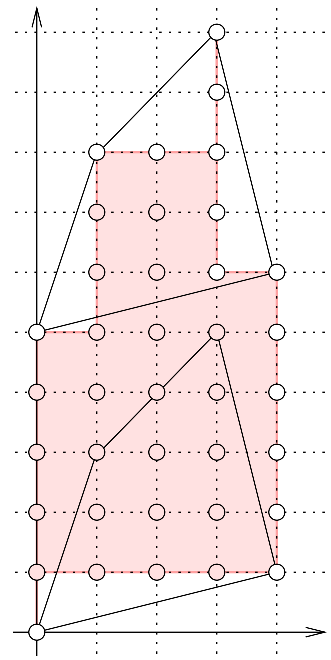

Figure 3: Support Hull of sum of projections

6.4 *Bound using Support Projections*

In general, the Dixon matrix size is sensitive to the variable order used in the construction. Thus, projections along different direction need to be considered more carefully; instead of approximating the size of the Dixon matrix by the sum of projections, it is possible to get a much tighter bound that can be shown to be exact for many non-trivial cases of supports. In fact, the results below are the first to show the dependence of the Dixon matrix construction on different variable orders.

We define the following projection operations on an arbitrary cornered support $\mathcal{P}$ with respect to a given variable order specified as $(l_1, \cdots, l_i)$, where $b = (b_1, \cdots, b_d)$, where b_j is the maximum value of the j-th component of any $\alpha \in \mathcal{P}$

$$\mathcal{P}_{(l_1,\ldots,l_i)} = \Big\{\, \alpha' = (\alpha'_1,\ldots,\alpha'_d) \,\Big|\, \alpha'_j = \alpha_j \text{ if } j \in \{l_1,\ldots,l_i\}, \\ \text{and } \alpha'_j = 0 \text{ otherwise, for } \alpha \in \mathcal{P} \Big\}.$$

For example, if $d = 4$, then $\mathcal{P}_{(1,4)} = \{(\alpha_1, 0, 0, \alpha_4) \mid \text{for all } \alpha \in \mathcal{P}\}$. This allows modeling of various variable orders. $\mathcal{P}_{(1,4)}$ notes that the first and fourth variables occur before the second and third, implying that the possible variable orders are (x_1, x_4, x_2, x_3), (x_1, x_4, x_3, x_2), (x_4, x_1, x_2, x_3), and (x_4, x_1, x_3, x_2). Hence $\pi_i(\mathcal{P}) = \mathcal{P}_{(l_{i+1},\ldots,l_d)}$ and $\mathcal{P}_{(1\ldots d)} = \mathcal{P}$. Let

$$\mathcal{P}_{(l_1\ldots l_i,*)} = \Big\{\, \alpha' = (\alpha'_1,\ldots,\alpha'_d) \,\Big|\, \alpha'_j = \alpha_j \text{ if } j \in \{l_1,\ldots,l_i\}, \\ \text{and } 0 \le \alpha'_j \le b_j \text{ otherwise, for } \alpha \in \mathcal{P} \Big\},$$

that is, the coordinates which are not specified, can assume any value within the range of the bounding box as determined by b.

For convenience, we will also assume below that a given support $\mathcal{A}$ is equal to its support hull, i.e., all support interior points with respect to the support hull of $\mathcal{A}$ are in $\mathcal{A}$.

Let us analyze how different $\mathcal{A}$ is from $\mathcal{B}$ for each possible variable order. Consider the complement of $\mathcal{A}$ wrt $\mathcal{B}$. This difference between $\mathcal{B}$ and $\mathcal{A}$ is analyzed by considering their successive projections along various coordinates. Let

$$\mathcal{C}_{(l_1\ldots l_i)} = \mathcal{B}_{(l_1\ldots l_i)} - \mathcal{A}_{(l_1\ldots l_i)}.$$

Note in the bivariate case, $\mathcal{C}_{(1,2)}$ is exactly the support complement. It should be noted that $\mathcal{C}_{(i)} = \oslash$ for any $i \in \{1,\ldots,d\}$.

For example, consider Figure 4, where a 3-dimensional support $\mathcal{A}$ and its two coordinate projections $\mathcal{A}_{(1,2)}$ and $\mathcal{A}_{(1,3)}$ are shown. By selecting the coordinates, a variable order used in the Dixon construction can be modeled. If $\mathcal{A}_{(1,2)}$ is considered, then the variable order is $[x, y, z]$ or $[y, x, z]$; in case $\mathcal{A}_{(1,3)}$ is considered, the variable order is $[x, z, y]$ or $[z, x, y]$.

The set $\mathcal{C}_{(l_1,\ldots,l_i)}$ is the difference of the bounding box $\mathcal{B}_{(l_1,\ldots,l_i)}$ and $\mathcal{A}_{(l_1,\ldots,l_i)}$ in $l_1,\ldots,l_i$-dimensions. For instance, $\mathcal{C}_{(1,2)}$ in Figure 4 has two points $(3,3)$ and $(3,4)$ corresponding to the variable orders $[x, y, z]$ or $[y, x, z]$; in case the variable order $[x, z, y]$ or $[z, x, y]$ is used, then $\mathcal{C}_{(1,3)}$ has only one point, $(3,3)$. Therefore, the choice of a variable order is important in analyzing the complement of $\mathcal{A}$ with respect to $\mathcal{B}$.

Just in the bivariate case, the support complement is defined to obtain an upper bound on the size of the Dixon polynomial.

Definition 19 *Given a support $\mathcal{A}$ and a list of coordinates $(l_1, \ldots, l_i)$, its* ***support complement***

$$S_i = \mathcal{C}_{(l_1 \ldots l_i)} - \mathcal{C}_{(l_1 \ldots l_{i-1},*)};$$

further,

$$S_{(i,*)} = \mathcal{C}_{(l_1 \ldots l_i,*)} - \mathcal{C}_{(l_1 \ldots l_{i-1},*)}.$$

Abusing the notation in the above, $\mathcal{C}_{(l_1 \ldots l_{i-1},*)} = \mathcal{B}_{(l_1 \ldots l_{i-1},*)} - \mathcal{A}_{(l_1 \ldots l_{i-1},*)}$.

It can be seen that $S_i \subseteq S_{(i,*)}$; $S_{(i,*)}$ includes all the points from the bounding box, whose first i coordinates, match the first i coordinates of any point from S_i.

Note that the definition of S_i is always with respect to some $l_1, \ldots, l_i$. Also, $\mathcal{C}_{(1,2)} = S_2 = S$ since $\mathcal{C}_{(1)} = \mathcal{C}_{(2)} = \oslash$ (thus, the notation is consistent with the notation used in Section 5).

We will call each set (S_i) for $i = 1, \ldots, d$, the i^{th}, a **support complement**. Informally, S_i attempts to include only those points of $\mathcal{C}_{(l_1 \ldots l_i)}$ which are not already included in $S_{(j,*)}, j < i$.

For example, in Figure 4, $\mathcal{C}_{(1,2,3)}$ consists of 14 points which can be identified from the figure; similarly, $\mathcal{C}_{(1,2,*)}$ can also be computed. Thus, $S_3 = \mathcal{C}_{(1,2,3)} - \mathcal{C}_{(1,2,*)}$ contains the following 6 points if the variable order $[x, y, z]$ is used: $(3,0,3)$, $(3,1,3)$ and $(0,3,2), (0,4,2), (0,3,3), (0,4,3)$. Note $S_{(2,*)} = \mathcal{C}_{(1,2,*)}$ (as $\mathcal{C}_{(1)} = B_{(1)} - \mathcal{A}_{(1)} = \oslash$ and $\mathcal{C}_{(1,*)} = \oslash$) is composed of 8 points which are $(3,3,*), (3,4,*)$.

Proposition 20 *Sets S_i, $S_{(i,*)}$ and $\mathcal{C}_{(l_1,\ldots,l_i)}$, as defined above, satisfy the following properties,*

(1) $|S_1| = 0$, *for any* $l_1, \ldots, l_d \in \{1, \ldots, d\}$.
(2) For $j < i$, $\mathcal{C}_{(l_1,\ldots,l_j,*)} \subseteq \mathcal{C}_{(l_1,\ldots,l_i,*)}$.
(3) $S_{(i,*)} \subseteq \mathcal{C}_{(l_1,\ldots,l_i,*)}$.
(4) $S_{(i,*)} \cap S_{(j,*)} = \oslash$ *for* $i \neq j$.

PROOF. 1. Since $\mathcal{C}_{(l_i)} = \oslash$, as $\mathcal{B}_{(l_i)} = \mathcal{A}_{(l_i)}$ for any $l_i \in \{1, \ldots, d\}$.

2. By the definition, if $p \in \mathcal{C}_{(l_1 \ldots l_j,*)}$, then $p \in \mathcal{B}_{(l_1 \ldots l_j,*)} = \mathcal{B}$ and $p \notin \mathcal{A}_{(l_1 \ldots l_j,*)}$. If $p \notin \mathcal{A}_{(l_1 \ldots l_j,*)}$, then $p \notin \mathcal{A}_{(l_1 \ldots l_i,*)}$ for $j < i$; $p \in \mathcal{C}_{(l_1 \ldots l_i,*)}$.

3. This is an immediate consequence of the definition, $S_{(i,*)} = \mathcal{C}_{(l_1 \ldots l_i,*)} - \mathcal{C}_{(l_1 \ldots l_{i-1},*)}$.

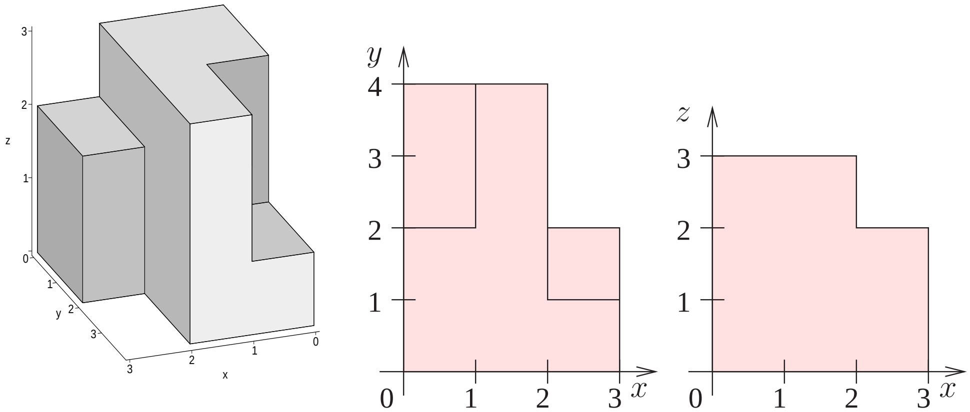

Fig. 4. A support $\mathcal{A}$ and its $\{x, y\}$ and $\{x, z\}$ projections.

4. Assume $j < i$. Since $S_i = \mathcal{C}_{(l_1 \dots l_i)} - \mathcal{C}_{(l_1 \dots l_{i-1}, *)}$, $S_{(i,*)} \not\subseteq \mathcal{C}_{(l_1 \dots l_{i-1}, *)}$. By 3 and 2 above, $S_{(j,*)} \subseteq \mathcal{C}_{(l_1 \dots l_j)} \subseteq \mathcal{C}_{(l_1 \dots l_{i-1}, *)}$, which implies that $S_{(i,*)} \cap S_{(j,*)} = \oslash$.

The following proposition relates a support $\mathcal{A}$ with its bounding box $\mathcal{B}$; it establishes that $\mathcal{A}$ is cut out from $\mathcal{B}$ by all S_i's.

Proposition 21 *Given a support hull $\mathcal{A}$, its bounding box $\mathcal{B}$, and $S_{(i,*)}$ as defined above, $i = 1, \dots, d$,*

$$\mathcal{A} = \mathcal{B} - \bigcup_{i=1}^{d} S_{(i,*)}.$$

PROOF. Clearly $S_{(i,*)} \subset \mathcal{B}$ and $S_{(i,*)} \cap \mathcal{A} = \oslash$. We only need to show that if $p \notin \mathcal{A}$ then $p \in S_{(i,*)}$ for some $i \in \{1, \dots, d\}$.

Since $p \notin \mathcal{A}$ then $p \in \mathcal{C}_{(l_1, \dots, l_d)}$. Then since

$$S_d = \mathcal{C}_{(l_1, \dots, l_d)} - \mathcal{C}_{(l_1, \dots, d-1, *)},$$

then $p \in S_d$ or $p \in \mathcal{C}_{(l_1, \dots, l_{d-1}, *)}$. In the former case we have same relationship where $p \in S_{d-1}$ or $p \in \mathcal{C}_{(l_1, \dots, l_{d-2}, *)}$. Since $S_{(1,*)} = \mathcal{C}_{(l_1, *)}$, p must belong to some $S_{(i,*)}$. $\square$

Sets $S_{(i,*)}$ are structured in such a way to model the projections. All the points in $S_{(i,*)}$ cannot be projected onto $S_{(i-1,*)}$. For each projection, only those points outside the support hull of $\mathcal{A}$ which cannot be projected onto a lower dimension need to be considered. Further, as in the bivariate case, these points are partitioned into disjoint sets which are located in the separate corners of $\mathcal{B}$.

6.4.1 Splitting Support Complement

Definition 22 *For every $k \in \mathbb{Z}_2^i$,*

$$S_i^k = \left\{ p \mid p \in S_i \quad and \quad \nexists\, \alpha \in \mathcal{A}_{(l_1 \dots l_i)} \quad s.t. \quad p \underset{k}{\leq} \alpha \right\}. \qquad (2)$$

In Figure 5, for $k = (0,1,1)$, $S_3^k = \{(0,0,3),(0,1,3)\}$; for $k = (1,1,1)$, $S_3^k = \{(3,2,2),(3,2,3),(3,3,2),(3,3,3)\}$. For all other $k \in \mathbb{Z}^3$, $S_3^k = \oslash$. There is only one part for S_2 when $k = (0,1)$; $S_2 = S_2^{(0,1)} = \{(0,2),(0,3)\}$. Hence, $S_{(2,*)} = \{(0,2,0),(0,3,0),(0,2,1),(0,3,1),(0,2,2),(0,3,2),(0,2,3),(0,3,3)\}$ (hollow points in Figure 5).

Proposition 23 *For any $i \in \{1, \dots, d\}$,*

$$S_i = \bigcup_{k \in \mathbb{Z}_2^i} S_i^k \quad but \quad |S_i| \leq \sum_{k \in \mathbb{Z}_2^i} |S_i^k|,$$

that is, S_i^k's are not necessarily disjoint for different k's.

PROOF. Let $p \in S_i$, then $p \in \mathcal{B}_{(l_1 \dots l_i)} - \mathcal{A}_{(l_1 \dots l_i)}$ and $p \notin \mathcal{A}_{(l_1 \dots l_i)}$. Therefore, there exists $k \in \mathbb{Z}_2^i$ such that $\forall\, \alpha \in \mathcal{A}_{1 \dots i}$, $p \underset{k}{\not\leq} \alpha$; by Definition 22, $p \in S_i^k$. $\square$

The case when S_i^k are not disjoint can be easily seen in the bivariate case. Figure 2 shows such a case when $S_2^{(1,0)}$ has an nonempty intersection with $S^{(0,1)}$.

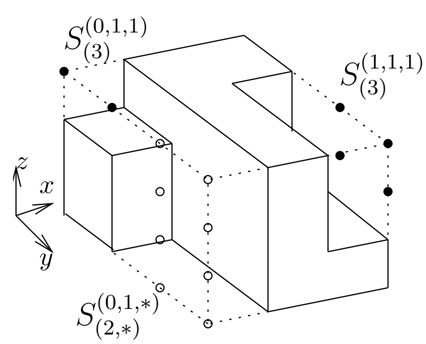

Figure 5: Sets $S_{(i,*)}^k$.

6.4.2 Relating Size of Support Complement to Size of Dixon Matrix

As $\mathcal{A}$ can be obtained from $\mathcal{B}$ using support complements $S_{(i,*)}$ for different i's, $\Delta_{\mathcal{A}}$ can be obtained from $\Delta_{\mathcal{B}}$ in terms $S_{(i,*)}$'s as shown below. This is similar to the analysis done for the bivariate case in [2].

Definition 24 *For any $k \in \mathbb{Z}_2^d$ and $r^k = (r_1^k, \dots, r_d^k) \in \mathbb{N}^d$, let*

$$T_i^k = r^k + S_i^k \quad and \quad T_i = \bigcup_{k \in \mathbb{Z}_2^d} T_i^k \quad where \quad r_j^k = \begin{cases} (j-1)\,b_j - 1 & if \ k_j = 1 \\ 0 & if \ k_j = 0 \end{cases}.$$

From T_i^k, define $T_{(i,*)}^k$ as was done above for $S_i^k, S_{(i,*)}^k$; and $T_{(i,*)} = \bigcup T_{(i,*)}^k$, where "*" is w.r.t. $\Delta_{\mathcal{B}}$. It is important to observe that the lattice points in $T_{(i,*)}$ are not in $\Delta_{\mathcal{A}}$.

Proposition 25 *Given a generic unmixed polynomial system a support $\mathcal{A}$, let $\mathcal{B}$ be its bounding box support. For any $i \in \{1, \ldots, d\}$,*

$$T_{(i,*)} \subseteq \Delta_{\mathcal{B}} - \Delta_{\mathcal{A}}.$$

PROOF. See [7] and [19].

This gives a set much smaller than $\Delta_{\mathcal{B}}$ in which the support $\Delta_{\mathcal{A}}$ of the Dixon polynomial is contained.

Theorem 26

$$\Delta_{\mathcal{A}} \subseteq \Delta_{\mathcal{B}} - \bigcup_{i=1}^{d} T_{(i,*)}.$$

PROOF. Since $\Delta_{\mathcal{A}} \subseteq \Delta_{\mathcal{B}}$ and if $p \in \Delta_{\mathcal{A}}$, then $p \notin T_i$ by Proposition 25. □

As shown in [2], the above relation becomes equality for $d = 2$. In general, the relation is that of a subset. An example in Figure 4 is a case where $\Delta_{\mathcal{A}}$ is strictly a subset. This is evident from the analysis of the Minkowski sum of projections since T_i's do not properly account for the difference between $\Delta_{\mathcal{B}}$ and $\Delta_{\mathcal{A}}$.

Later in the paper, conditions on a support are developed for which the above relation becomes equality.

To estimate an upper bound on the size of the support of the Dixon polynomial and the size of the Dixon matrix, the following properties of T_i's are relevant.

Proposition 27 *For $i, j \in \{0, \ldots, d\}$ s.t. $i \neq j$ and $k, l \in \mathbb{Z}_2^i$ such that $k \neq l$,*

(i) $T_i = \bigcup_{k \in \mathbb{Z}_2^i} T_i^k$,
(ii) $\left| T_i^k \right| = \left| S_i^k \right|$,
(iii) $T_i^k \cap T_i^l = \oslash$,

(iv) $T_{(i,*)} \cap T_{(j,*)} = \oslash$,

PROOF. Statements (i) and (ii) follow from Definition 24.

Statement (iii): The proof is done by contradiction. Suppose there exists $p \in T_i^k \cap T_i^l$. By Definition 24,

$$s + r^k = p = q + r^l \quad \text{for} \quad s \in S_i^k \quad \text{and} \quad q \in S_i^l.$$

Since $k \neq l$, let j be the smallest integer such that $k_j \neq l_j$; w.l.o.g. let $k_j = 0$ and $l_j = 1$. Thus,

$$s_j = p_j = q_j + \underbrace{(j-1)b_j - 1}_{r^l}, \tag{3}$$

where b_j is the maximum value of the j^{th} coordinate for all points in $\mathcal{A}$. Since $s \in S_i^k$, there exists an $\alpha \in \mathcal{A}$ such that $s_j < \alpha_j \leq b_j$; the above equality might hold only for $j = 1$ or $j = 2$.

If k and l disagree only on a single coordinate j, then $0 \leq s_j < \alpha_j < q_j \leq b_j$ since $l_j = 0$ by assumption, making the equality (3) impossible. If k and l disagree on first two coordinates, there are a few cases. Assuming that $k_1 = 0$, $l_1 = 1$,

$$s_1 = q_1 - 1 \quad \text{and} \quad \begin{cases} s_2 = q_2 + b_2 - 1 \text{ if } k_2 = 0, \\ s_2 + b_2 - 1 = q_2 \text{ if } k_2 = 1. \end{cases}$$

However, if $k_2 = 0$ then $q_2 = 0$ as $s_2 < b_2$. Since $l_2 = 1$, it would contradict the fact that $q \in S_i^l$; on the other hand if $k_2 = 1$ and $l_2 = 0$, then at best $q_2 = b_2$ and $s_2 = 1$ contradicting that $q \in S_i^l$ and $s \in S_i^k$. Therefore, there is no such p in $T_i^k \cap T_i^l$; hence, $T_i^k \cap T_i^l = \oslash$.

Statement (iv): The proof is by contradiction. Suppose $T_{(i,*)} \cap T_{(j,*)} \neq \oslash$ and $p \in T_{(i,*)} \cap T_{(j,*)}$. Then, $p \in T_{(i,*)}^k$ and $p \in T_{(j,*)}^l$ for some $k, l \in \mathbb{Z}_2^d$. By Definition 24,

$$p = r^k + s^k \quad \text{for some} \quad s^k \in S_i^k \qquad \text{and} \qquad p = r^l + s^l \quad \text{for some} \quad s^l \in S_j^l.$$

Note that $k \neq l$ since $S_i \cap S_j = \oslash$.

Assume w.l.o.g. that $k_0 \neq l_0$ and $k_0 = 0$ where $l_0 = 1$. Then there exists $\alpha \in \mathcal{A}$ such that

$$0 \leq s_0^k < \alpha_0 < s_0^l \leq b_0;$$

thus, $p_0 = s_0^k < s_0^l$, contradicting that $p_0 = r_0^l + s_0^l$. Therefore, there is no $p \in T_{(i,*)} \cap T_{(j,*)}$; hence, $T_{(i,*)} \cap T_{(j,*)} = \oslash$. $\square$

Using Proposition 27 (i), (ii), and (iii), it follows that

$$|T_i| = \sum_{k\in\mathbb{Z}_2^i} |S_i^k|.$$

Using the above properties of T_i, the size of its intersection with $\Delta_{\mathcal{B}}$ can be estimated.

Proposition 28 *Assuming a coordinate order* $(l_1, \ldots, l_d)$*,*

$$\left|T_{(i,*)} \cap \Delta_{\mathcal{B}}\right| = \frac{d!}{i!}\,|T_i| \prod_{j=i+1}^{d} b_{l_j}.$$

PROOF. $T_i \subset \Delta_{\mathcal{B}} = \{\, p \mid 0 \le p_i < i\, b_i\}$. Thus,

$$\left|\Delta_{\mathcal{B}(l_{i+1}\ldots l_d)}\right| = \frac{d!}{i!} \prod_{j=i+1}^{d} b_{l_j}. \qquad \square$$

Using Propositions 17, 27 and 28, an upper bound on the size of the support of the Dixon polynomial can be derived. This also gives an upper bound on the size of the Dixon matrix and in turn, the degree of the projection operator. Here is the main result:

Theorem 29

$$\left|\Delta_{\mathcal{B}} - \bigcup_{i=1}^{d} T_{(i,*)}\right| = d! \prod_{i=1}^{d} b_i - \sum_{i=1}^{d} \frac{d!}{i!} |T_i| \prod_{j=i+1}^{d} b_j.$$

And, an upper bound on the size of the Dixon polynomial is given as:

$$|\Delta_{\mathcal{A}}| \le d! \prod_{i=1}^{d} b_i - \sum_{i=1}^{d} \left(\frac{d!}{i!} |T_i| \prod_{j=i+1}^{d} b_{l_j} \right). \tag{4}$$

7 Multivariate Case: Corner-Cut and Almost Corner-Cut Supports

The above inequality (4) can be used to identify a class of unmixed polynomial systems whose support is such that the matrices constructed using the Dixon formulation are exact, i.e., the size coincides with the BKK bound. In other words, cases where

$$|\Delta_{\mathcal{A}}| = d\,! \prod_{i=1}^{d} b_i - \sum_{i=1}^{d} \left(\frac{d!}{i!} \; |T_i| \prod_{j=i+1}^{d} b_j \right) \qquad \Longrightarrow \qquad |\Delta_{\mathcal{A}}| = d\,! \; \mathsf{Vol}(\mathcal{A}).$$

For supports for which (4) becomes equality, the Dixon formulation produces exact resultants. Such supports include d-degree systems [17] for the general multivariate case and the bivariate corner-cut supports [5,4,2] For example, for a d-degree system, all sets $S_i = \oslash$ and $|T_i| = 0$. Thus, $|\Delta_{\mathcal{A}}| = |\Delta_{\mathcal{B}}|$ and $|\Delta_{\mathcal{A}}| = d\,! \prod_{i=1}^{d} b_i = d\,! \, \mathsf{Vol}(\mathcal{A})$. This result thus generalizes most known results about unmixed polynomial systems for which resultants can be computed exactly. [3]

We give below a generalization of the concept of a bivariate corner-cut support introduced in [5,4] for which the Dixon based resultant methods compute the exact resultant.

7.1 Corner-Cut Supports in d-Dimension

A support $\mathcal{A}$ is called *corner-cut* if and only if for every coordinate order $(l_1, \ldots, l_d)$, the following conditions are satisfied:

(1) the projection $\mathcal{A}_{(l_1 \ldots l_{d-1})} = \mathcal{B}_{(l_1 \ldots l_{d-1})}$, and
(2) for each $k \in \mathbb{Z}_2^d$, S_d^k (by Definition 22) is a d-dimensional rectangle.

Figure 6 shows an example of a 3-D corner-cut support. For any variable order, if the last coordinate is dropped, a rectangular support is obtained, i.e. $\mathcal{A}_{(1,2)} = \mathcal{B}_{(1,2)}$ for any order, thus satisfying the first condition.

The set $\mathcal{B} - \mathcal{A}$ is composed of the union of rectangular regions, each appearing in some corner of $\mathcal{B}$. Each of those rectangles is S_d^k for various values of $k \in \mathbb{Z}_2^3$, thus satisfying the second condition.

In the bivariate case, it is always the case that $\mathcal{A}_{(l_1)} = \mathcal{B}_{(l_1)}$. Thus, the first condition is trivially true. The above definition of a corner-cut support is a generalization of a bivariate corner-cut support introduced in [5,4] and studied in [2] to higher dimensions. As discussed earlier, an obvious generalization does not work (see Figure 4). In that sense, we have settled open problems posed in [5,4].

[3] Nevertheless, there are other systems including multi-graded systems [6,20], which cannot be generalized this way.

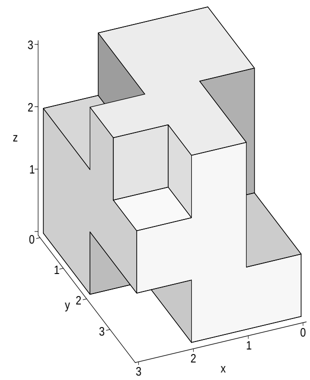

Fig. 6. 3D corner-cut support support.

Below, we prove that if a support is corner-cut, the size of the Dixon polynomial as well as the degree of the associated projection operator equal the BKK bound, which is also the degree of its toric resultant. Therefore, the projection operator extracted from the Dixon matrix is precisely the resultant of a given polynomial system.

Theorem 30 (d-dimensional Corner-Cut) *Given a generic unmixed polynomial system $\mathcal{F}$ with a* corner-cut *support $\mathcal{A}$, the Dixon Matrix is exact for any variable order used to construct it.*

PROOF. It is proved that

$$d\,!\ \mathsf{Vol}(\mathcal{A}) = d\,!\prod_{i=1}^{d} b_i - \sum_{i=1}^{d}\left(\frac{d!}{i!}\ |T_i| \prod_{j=i+1}^{d} b_{l_j}\right) = d\,!\prod_{i=1}^{d} b_i - |T_d|,$$

The second equality above is implied since $\mathcal{A}$ is corner-cut in which $T_i = \oslash$ for all $1 \leq i < d$; this implies that $|\Delta_{\mathcal{A}}| = d\,!\,\mathsf{Vol}(\mathcal{A})$. Because of the corner-cut condition, the variable order does not change the upper bound.

For $k \in \mathbb{Z}_2^d$, let $b^k = (b'_1, \ldots, b'_d)$ where $b'_i = b_i$ if $k_i = 1$ and $b'_i = 0$ otherwise, that is, let b^k be the point in k^{th} corner of the bounding box. Consider the convex hull complement of $\mathcal{A}$ and its partition

$$Q = \mathsf{Conv}(\mathcal{B}) - \mathsf{Conv}(\mathcal{A}), \qquad \text{and} \qquad Q^k = \{\, q \mid q \in Q \quad \text{and line } [b^k, q] \subset Q \,\}.$$

This set was used for a bivariate corner-cut support in [2]; here we consider its generalization to the multivariate case. Because $\mathcal{A}$ is corner-cut, for $k, l \in \mathbb{Z}_2^d$ and $k \neq l$,

$$Q^k \cup Q^l = \oslash \qquad \text{and} \qquad Q = \bigcup_{k \in \mathbb{Z}_2^d} Q^k.$$

Since

$$|T_d| = \sum_{k \in \mathbb{Z}_2^d} |S_d^k|,$$

it can be proved that

$$|S_d^k| = d!\, \mathsf{Vol}(Q^k),$$

from which the statement of the theorem follows.

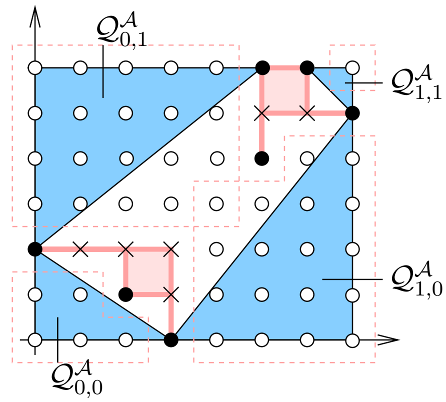

Figure 7: Newton polytope Complement.

Since each S_d^k is rectangular, the size of

$$|S_d^k| = \prod_{i=1}^{d} s_i,$$

where s_i is the number of points of S_d^k along the i^{th} coordinate.

But Q^k is a corner simplex whose sides are of length s_i; hence, its volume is

$$\mathsf{Vol}(Q^k) = \frac{1}{d!} \prod_{i=1}^{d} s_i;$$

therefore, $d!\, \mathsf{Vol}(Q^k) = |S_d^k|$. Thus,

$$\begin{aligned} d!\, \mathsf{Vol}(\mathcal{A}) &= d!\, \mathsf{Vol}(\mathcal{B}) - d!\, \mathsf{Vol}(Q) = |\Delta_{\mathcal{B}}| - \sum_{k \in \mathbb{Z}^d} d!\, \mathsf{Vol}(Q^k) \\ &= |\Delta_{\mathcal{B}}| - \sum_{k \in \mathbb{Z}^d} |S_d^k| = |\Delta_{\mathcal{B}}| - \sum_{k \in \mathbb{Z}^d} |T_d^k| = \left| \Delta_{\mathcal{B}} - \bigcup_{k \in \mathbb{Z}^d} T_d^k \right| \\ &= |\Delta_{\mathcal{B}} - T_d|\,. \end{aligned}$$

Since by Theorem 29, $d!\, \mathsf{Vol}(\mathcal{A}) \le |\Delta_{\mathcal{A}}| \le |\Delta_{\mathcal{B}} - T_d|$, it follows that $d!\, \mathsf{Vol}(\mathcal{A}) = |\Delta_{\mathcal{A}}|$. This implies that the Dixon matrix is exact. □

7.2 Almost Corner-Cut Supports in d-Dimension

The above notion of a corner-cut support is a proper generalization in d-dimension of a similar notion introduced for the bivariate case, where only rectangular regions from each corner are absent. In the multivariate case, it is

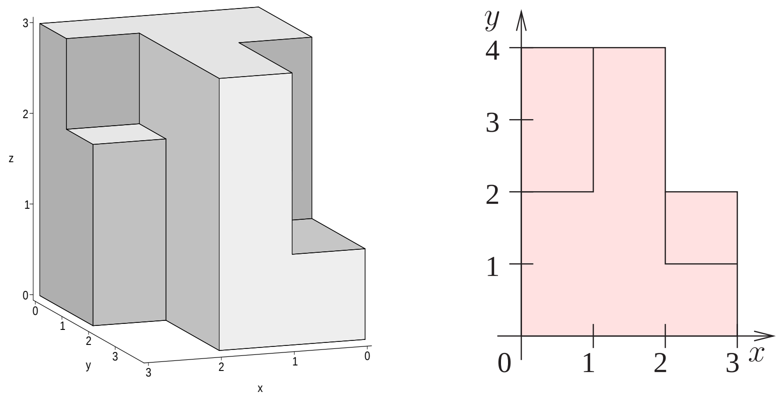

Fig. 8. Almost corner-cut support $\mathcal{A}$ and its projection $\mathcal{A}_{(1,2)}$.

important that these rectangular regions do not overlap for adjacent corners. There are however supports, which are not corner-cut, but happen to be very similar to a corner-cut and for which the Dixon matrices are exact.

A support $\mathcal{A}$ is called *almost corner-cut* if and only if the following conditions are satisfied:

(1) There exist **a unique fixed** coordinate l_d, $1 \leq l_d \leq d$, (the corresponding variable is chosen to be substituted the last in the construction) such that for all coordinate orders $(l_1, \ldots, l_{d-1}, l_d)$, in which x_{l_d} is the last variable in the variable order, possibly $\mathcal{A}_{(l_1 \ldots l_{d-1})} \neq \mathcal{B}_{(l_1 \ldots l_{d-1})}$.
(2) for each $k \in \mathbb{Z}_2^d$, S_d^k is a d-dimensional rectangle.
(3) for each $k \in \mathbb{Z}_2^{d-1}$, S_{d-1}^k is a $d-1$ dimensional rectangle, where the coordinate order is fixed as in condition (1).

For example, the support in Figure 8 is not corner-cut but is almost corner-cut. The example in Figure 4 for instance is neither corner-cut nor almost corner-cut: there are two choices for the last coordinate y and z for which $\mathcal{A}_{(l_1 \ldots l_{d-1})} \neq \mathcal{B}_{(l_1 \ldots l_{d-1})}$.

The proof technique used to show that the Dixon formulation computes the resultant exactly for an almost corner-cut support is patterned after the proof for that of a corner-cut support.

Theorem 31 *Given a generic unmixed polynomial system $\mathcal{F}$ with an* almost corner-cut *support $\mathcal{A}$, the Dixon Matrix is exact for every variable order in which the last coordinate satisfies the properties the definition of an almost corner-cut support.*

PROOF. It is shown below that

$$d\,!\ \mathsf{Vol}(\mathcal{A}) = d\,!\prod_{i=1}^{d} b_i - \sum_{i=1}^{d}\left(\frac{d!}{i!}\ |T_i|\ \prod_{j=i+1}^{d} b_{l_j}\right).$$

Since $T_i = \oslash$ for $i < d-1$, the above becomes

$$d\,!\ \mathsf{Vol}(\mathcal{A}) = d\,!\prod_{i=1}^{d} b_i - |T_d| - d\,b_{l_d}\,|T_{d-1}|.$$

Let Q and Q^k be as in the proof of Theorem 30. In this case,

$$Q = \bigcup_{k\in\mathbb{Z}_2^d} Q^k, \qquad \text{but not necessarily} \qquad Q^k \cap Q^l = \oslash,$$

for $k, l \in \mathbb{Z}_2^d$ and $k \neq l$.

If $Q^k \cap Q^l \neq \oslash$, then $k_i = l_i$ for all $i \neq d$. So let $k' = (k_1, \ldots, k_{d-1})$ and note that both b^k and b^l are in $S^{k'}_{(d-1,*)}$. In general,

$$\begin{aligned}\mathsf{Vol}(Q^k \cup Q^l) = &\Big(\mathsf{Vol}(Q^k) - \mathsf{Vol}(Q^k \cap Q^l)\Big) + \Big(\mathsf{Vol}(Q^l) - \mathsf{Vol}(Q^k \cap Q^l)\Big)\\ &+ \mathsf{Vol}(Q^k \cap Q^l),\end{aligned}$$

but

$$d\,!\ \Big(\mathsf{Vol}(Q^k) - \mathsf{Vol}(Q^k \cap Q^l)\Big) = |S^k_{d-1}|$$

and

$$d\,!\ \Big(\mathsf{Vol}(Q^l) - \mathsf{Vol}(Q^k \cap Q^l)\Big) = |S^l_d|,$$

and also

$$d!\mathsf{Vol}(Q^k \cap Q^l) = b_{l_d}\ |S^{k'}_{d-1}|.$$

which can be verified in the same manner as in Theorem 30.

Hence, all the volume "missing" from $\mathcal{B}$, which is the volume of Q, equals $|T_d| + b_{l_d}\,|T_{d-1}|$. So,

$$\begin{aligned}d\,!\ \mathsf{Vol}(\mathcal{A}) &= d\,!\ \mathsf{Vol}(\mathcal{B}) - d\,!\ \mathsf{Vol}(Q)\\ &= |\Delta_{\mathcal{B}}| - \sum_{k\in\mathbb{Z}^d} |S^k_d| - d\sum_{k'\in\mathbb{Z}^{d-1}} b_{l_d}|S^{k'}_d|\\ &= |\Delta_{\mathcal{B}}| - |T_d| - d\,b_{l_d}\,|T_{d-1}|.\end{aligned}$$

Since the upper and lower bounds for $|\Delta_{\mathcal{A}}|$ are the same, it follows that $d\,!\ \mathsf{Vol}(\mathcal{A}) = |\Delta_{\mathcal{A}}|$, implying that the Dixon-based methods compute the resultant in this case exactly. □

A reader might be interested in finding why the above argument does not work for the general case. It seems that there does not exist one-to-one correspondence between S_i's and T_j's: T_j's depend on the projection chosen where as S_i's do not. A complete analysis of this relationship has to consider the dependence of T_j's on the variable order chosen.

We have thus settled an open problem raised in [5] of generalizing a bivariate corner-cut support to the general multidimensional case. It is proved above that the Dixon-based resultant methods compute resultants of generic unmixed polynomial systems if their supports are either corner-cut or almost corner-cut.

There are however families of generic unmixed polynomial systems whose support is neither corner-cut nor almost corner-cut, yet the Dixon formulation still produces exact resultants. A notable family of such polynomial systems is that of multi-graded systems introduced in [21] and analyzed for the Dixon construction in [6]. It would, however, be interesting to have an example of a generic unmixed polynomial system whose support is not one of multi-graded, corner-cut and almost corner-cut, but the Dixon formulation computes its resultant.

8 Conclusion

The paper generalizes the results in [5,4,2] for the bivariate case to the general multivariate case. Using the concepts of support-interior points and support hull of the support of a generic unmixed multivariate polynomial system, the concept of a d-dimensional corner-cut support is defined. It is proved that for generic unmixed polynomial systems with d-dimensional corner-cut supports, the Dixon-based resultant methods (both the generalized Dixon method as defined in [9] as well as the Dixon multiplier method defined in [22,23]) generate exact resultants. As a byproduct, the Dixon multiplier method also produces Sylvester-type resultant matrices for generic unmixed polynomial systems with d-dimensional corner-cut supports. Further, the variable ordering used in constructing Dixon-based resultant matrices does not affect the outcome, i.e., (exact) resultants are generated irrespective of the variable ordering. The complexity of the method is not affected by the chosen variable ordering either. This settles an open problem in [5].

It is also shown that these results can be generalized for generic unmixed polynomial systems with d-dimensional almost-corner-cut supports if certain variable orderings are chosen (these orderings only fix the last variable to be substituted in the construction).

Tighter bounds on the size of the Dixon matrix and on the degree of the projection operator extracted from it are proved. These bounds are based on analyzing how much a given support deviates from the support of an associated d-degree system for which the Dixon formulation computes the exact resultant. This improves upon the related bounds proved in [17,18]. As in the bivariate case, the size of the support of the Dixon polynomial of a given generic unmixed polynomial system is shown to be lower than or equal to that of an associated d-degree system minus the sum of all support hull complements.

From the above bound on the degree of a projection operator for a given unmixed polynomial system, an bound on the degree of the extraneous factor, if any, in the projection operator can be determined a priori. A projection operator extracted from the associated Dixon matrix can be factored; any factor of degree higher than the bound obtained on the extraneous factor is a part of the resultant. This information can thus lead to easier identification of the extraneous factor in a projection operator.

The above analysis also gives sharper bounds on the complexity of resultant computations based on the Dixon formulation in terms of its support since the complexity is governed by the determinant computations of Dixon matrices. Any deviation from a d-degree support is abstracted by the notion of the support complement, from which a lower bound on the deviation from the size of the support of the Dixon polynomial of a d-degree system is obtained.

The insight developed for defining almost-corner-cut supports is likely to be helpful in defining a heuristic for variable ordering for unmixed as well as mixed polynomial systems that computes projection operators with extraneous factors of lower degrees. A method for finding translation vectors as well as a term for constructing Dixon multiplier matrices is being investigated by generalizing the ideas developed in [2] for the bivariate case.

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