Resultants for Unmixed Bivariate Polynomial Systems using the Dixon formulation *

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Abstract

A necessary and sufficient condition on the support of a generic unmixed bivariate polynomial system is identified such that for polynomial systems with such support, the Dixon resultant formulation produces their resultants. It is shown that Sylvester-type matrices can also be obtained for such polynomial systems. These results are shown to be a generalization of related results recently reported by Chionh as well as Zhang and Goldman. For a support not satisfying the above condition, the degree of the extraneous factor in the projection operator computed by the Dixon formulation is calculated by analyzing how much the support deviates from a related rectangular support satisfying the condition. The concept of a *support interior* point of a support is introduced; a generic inclusion of terms corresponding to support interior points in a polynomial system is shown not to affect the degree of the projection operator computed by the Dixon construction.

For generic mixed bivariate systems, "good" Sylvester type matrices can be constructed by solving an optimization problem on their supports. The determinant of such a matrix gives a projection operator with a low degree extraneous factor. The results are illustrated on a variety of examples.

1 Introduction

New results characterizing generic unmixed polynomial systems with two variables for which resultants can be exactly computed and Sylvester-type matrices can be constructed, are proved. Earlier in [CK00a], Chtcherba and Kapur had shown that the support of a bivariate unmixed polynomial system not including *an orderable simplex* is a necessary and sufficient condition for the determinant of the associated Dixon matrix being exact resultant (without any extraneous factors). Independently, Chionh [Chi01], Zhang and Goldman [ZG00], as well as Zhang in his Ph.D. thesis [Zha00] proposed *corner-cut* supports for which Dixon matrices as well as Sylvester-type multiplier matrices can be constructed whose determinant is the exact resultant. The results in this paper are shown to be related to and more general than those in [Chi01, ZG00, Zha00]. A necessary and sufficient condition on bivariate supports is identified such that for a generic unmixed polynomial system with such a support, its resultant can be computed exactly using constructions based on the Dixon resultant formulation. Such bivariate supports are shown to include Chionh's supports as well as Zhang and Goldman's corner-cut supports for which they proved that resultants can be computed exactly. In addition, for bivariate polynomial systems whose support does not satisfy these conditions, the proposed construction estimates the degree of the extraneous factor in a projection operator computed from a Dixon multiplier matrix.

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The algorithm for constructing multiplier matrices based on the Dixon resultant formulation works in general for polynomial system from which more than two variables need to be eliminated even when the polynomial system is not necessarily unmixed. These multiplier matrices can be used to extract (in most cases) the resultants as determinants of their maximal minors. The approach generalizes a related method for constructing multiplier matrices from the Dixon resultant formulation discussed in [CK00b]. Beside being a generalization, the approach has the advantage of generating Sylvester-like multiplier matrices whose determinants are resultants even in cases where the earlier method by Chtcherba and Kapur produces an extraneous factor.

It is also shown that for the bivariate case, the proposed construction produces multiplier matrices with resultants as their determinants even in some mixed systems. The supports of the polynomial system are translated so that they have an nonempty intersection and then a term in the nonempty intersection of translated supports is used for constructing the multiplier matrix. This is formulated as an optimization problem that minimizes the size of the Dixon multiplier matrix. The approach is compared with other approaches, and is shown to be more efficient and to work better on many examples of practical interest.

For unmixed polynomial systems in which more than two variables are simultaneously eliminated, the determinant of the associated Dixon multiplier matrix is shown to be not necessarily the resultant even for corner-cut supports. Nevertheless, preliminary results show that with proper generalization to polynomial systems with more variables, the results will hold when more than two variables are eliminated.

Section 2 defines supports of a polynomial and a polynomial system, and reviews the BKK bound for toric roots and toric resultants of a polynomial system. Section 3 reviews the Dixon formulation of resultants, where the Dixon matrix and the Dixon polynomial of a given polynomial system are introduced. Section 3.1 analyzes the support of the Dixon polynomial in terms of the support of the polynomial system. It is shown that the support of the Dixon polynomial (which determines the size of the Dixon matrix) can be expressed as a union of the support of the Dixon polynomials of polynomial systems corresponding to simplexes (a simplex support has three distinct points).

Section 4 is a detailed analysis of the support of the Dixon polynomial in relation to the support of unmixed polynomial systems. For a simplex support, the support of its Dixon polynomial is precisely characterized in terms of the *projection sum* expressed in terms of the coordinates of the simplex. Points inside the convex hull of the support of a polynomial system are classified into two categories: (i) *support interior* points such that when terms corresponding to these support points are included in the polynomial system, the support of the Dixon polynomial and hence, the size of the Dixon matrix does not change, (ii) other support points such that the corresponding terms when included in the polynomial system contribute to the extraneous factors in the projection operator computed from the associated Dixon matrix. Using these concepts, the notion of the *support hull* of a support is defined which includes along with the support, all its support interior points. Using the support hull of the support of an unmixed polynomial system, the support of its Dixon polynomial is precisely characterized using the projection sum of the support.

The concept of *support complement* characterizing how different a given support is from a bi-degree support (in the case of bivariate systems) is introduced; this support complement can be partitioned into four corners. It is shown that for a given polynomial system, the support of its Dixon polynomial can be shown to be a rectangle (constructed from the bounding bi-degree system) from which the four corner support complement determined from the support of the polynomial system are removed. Thus the size of the Dixon matrix (which is the cardinality of the support of the Dixon polynomial) can be precisely determined based on the size of the corners. It is also proved that if a term corresponding to a support interior point of a given support is generically included, the modified unmixed polynomial system will lead to the Dixon matrix of the same size as obtained from the original unmixed polynomial system.

Section 5 has one of the main results of the paper. It is shown that if the support of the polynomial system after inclusion of its support interior points is a rectangle with four rectangular corners removed, then the size of the Dixon matrix is the same as the BKK bound; this implies that for generic unmixed polynomial systems with such supports, the Dixon formulation computes the resultant exactly. In contrast, Chionh [Chi01] proved that the Dixon formulation computes the exact resultant for generic unmixed polynomial systems whose support is a rectangle with four rectangular corners removed.

Section 6 proves a result about the degree of the extraneous factor in the projection operator computed by the Dixon resultant formulation for generic unmixed polynomial systems whose supports do not satisfy the above-stated condition.

In Section 7, these results are extended to Dixon multiplier matrices, which are Sylvester type matrices but constructed using the Dixon formulation. Zhang and Goldman's results about corner cut supports are shown to be a special case of our results discussed in Sections 5 and 7. It is shown that an obvious generalization of corner-cut supports does not work even for trivariate polynomial systems. This is followed by the section discussing examples of unmixed systems, and a comparison of different approaches. The Dixon multiplier matrix method turns out to have many advantages over other methods for computing resultants.

Section 9 considers mixed polynomial systems. A heuristic to generate "good" Dixon multiplier matrices whose determinants are projection operators having extraneous factors of minimal degree, is discussed. This heuristic utilizes terms common in the supports of the mixed polynomial system for generating the Dixon multipler matrix. Supports are translated to maximize overlap among them. Determining how much supports ought to be translated as well as the term to be selected for generating the Dixon multiplier matrix can be formulated as an optimization problem, minimizing the support of the Dixon polynomial. An example illustrating this idea is discussed in detail. Section 10 compares our results experimentally with other approaches on examples of mixed polynomial systems.

Section 11 discusses issues for further investigation as well as possible generalization of these results to multivariate polynomial systems.

2 Bivariate Systems

Consider a bivariate polynomial system $\mathcal{F}$,

$$f_0 = \sum_{\alpha \in \mathcal{A}_0} a_\alpha \, x^{\alpha_x} y^{\alpha_y}, \qquad f_1 = \sum_{\beta \in \mathcal{A}_1} b_\beta \, x^{\beta_x} y^{\beta_y}, \qquad f_2 = \sum_{\gamma \in \mathcal{A}_2} c_\gamma \, x^{\gamma_x} y^{\gamma_y},$$

where for $i = 0, 1, 2$, each finite set $\mathcal{A}_i$ of nonnegative integer tuples is called the *support* of the polynomial f_i; further, $\alpha = \langle \alpha_x, \alpha_y \rangle$, $\beta = \langle \beta_x, \beta_y \rangle$, and $\gamma = \langle \gamma_x, \gamma_y \rangle$. If $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2$, the polynomial system is called *unmixed*; otherwise, it is called *mixed*.

The support of a polynomial system $\mathcal{F}$ is written as $\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle$. Given a support $\mathcal{A}_i$, let $\mathsf{Vol}(\mathcal{A}_i)$ stand for the Euclidean volume of the convex hull (Newton polytope) of $\mathcal{A}_i$.

Theorem 2.1 (BKK) *Given two bivariate polynomials f_1, f_2, with corresponding supports $\mathcal{A}_1$ and $\mathcal{A}_2$, the number of common toric roots of these polynomials is either infinite or at most*

$$\mu(\mathcal{A}_1, \mathcal{A}_2) = \mathsf{Vol}(\mathcal{A}_1 +^1 \mathcal{A}_2) - \mathsf{Vol}(\mathcal{A}_1) - \mathsf{Vol}(\mathcal{A}_2);$$

further, for most choices of coefficients, this bound is exact. The function μ is called the mixed volume function [GKZ94].

If $\mathcal{A}_1 = \mathcal{A}_2$, then $\mu(\mathcal{A}_1, \mathcal{A}_2) = 2\mathsf{Vol}(\mathcal{A}_1)$.

In general, a polynomial system is called *generic* if it has a finite number of roots which is maximal for any choice of coefficients. The polynomial system $\{f_1, f_2\}$ is thus generic if the number of toric roots of any two polynomials equals its BKK bound. If we assume that coefficients are algebraically independent, then the polynomial system is certainly generic. Henceforth, the coefficients of terms in a polynomial system are assumed to be algebraically independent, unless stated otherwise.

In a generic case, the toric resultant of $\mathcal{F} = \{f_0, f_1, f_2\}$ is of degree equal to the BKK bound based on any two polynomials, in terms of the coefficients of the remaining polynomial [PS93]. For example, the degree of the resultant in terms of coefficients of f_0 is $\mu(\mathcal{A}_1, \mathcal{A}_2)$.

Using the Sylvester dialytic method, one can construct the resultant matrix for a given polynomial system by multiplying each polynomial by a set of monomials, called its *multipliers*, and rewriting the resulting polynomial

[1] The sum $\mathcal{A}_1 + \mathcal{A}_2$ is the Minkowski sum of polytopes $\mathcal{A}_1, \mathcal{A}_2$, where $p \in \mathcal{A}_1 + \mathcal{A}_2$ if $p = q + r$ for $q \in \mathcal{A}_1$ and $r \in \mathcal{A}_2$ where $+$ is the regular vector addition; see [CLO98] for definitions.

system in the matrix notation. Let $X_i = \{\ x^a y^b\ \}$, $i = 0,1,2$, be the multiplier set for the polynomial f_i, respectively; then the matrix is constructed as

$$\begin{pmatrix} X_0 f_0 \\ X_1 f_1 \\ X_2 f_2 \end{pmatrix} = M \times X,$$

where X is the ordered set of all monomials appearing in $X_i f_i$ for $i = 0,1,2$. Note in order for M to qualify as a resultant matrix, $|X_0| \geq \mu_0 = \mu(\mathcal{A}_1, \mathcal{A}_2)$, $|X_1| \geq \mu_1 = \mu(\mathcal{A}_0, \mathcal{A}_2)$, and $|X_2| \geq \mu_2 = \mu(\mathcal{A}_0, \mathcal{A}_1)$.

If it can be shown that the matrix M above is square and non-singular, then it is the resultant matrix since the determinant of M ha to be a multiple of the resultant. Moreover, if $|X_i| = \mu_i$, then M is exact, in the sense that its determinant is exactly the resultant of $\mathcal{F} = \{f_0, f_1, f_2\}$.

3 The Dixon Resultant Matrix

In this section, we briefly review the generalized Dixon formulation, first introduced by Dixon [Dix08], and generalized by Kapur, Saxena and Yang [KSY94, KS96]. We will consider the bivariate case only.

Define the *Dixon polynomial* to be

$$\theta_{x,y}(f_0, f_1, f_2) = \frac{1}{(\overline{x} - x)(\overline{y} - y)} \begin{vmatrix} f_0(x,y) & f_1(x,y) & f_2(x,y) \\ f_0(\overline{x},y) & f_1(\overline{x},y) & f_2(\overline{x},y) \\ f_0(\overline{x},\overline{y}) & f_1(\overline{x},\overline{y}) & f_2(\overline{x},\overline{y}) \end{vmatrix}, \qquad (1)$$

where $\overline{x}$ and $\overline{y}$ are new variables and for each $0 \leq i \leq 2$, $f_i(\overline{x}, y)$ is the polynomial obtained by replacing x in $f_i(x,y)$ by $\overline{x}$; polynomials $f_i(\overline{x}, \overline{y})$ are similarly defined. Let $\overline{X}$ be an ordered set of all monomials appearing in $\theta(f_0, f_1, f_2)$ in terms of variables $\overline{x}$, $\overline{y}$, and X be the set of all monomial in terms of variables x and y. Then

$$\theta_{x,y}(f_0, f_1, f_2) = \overline{X}\ \Theta_{x,y}\ X,$$

where $\Theta_{x,y}$ is called the *Dixon matrix*. Note that $\Theta_{x,y} = \Theta^T_{y,x}$, where the order of variables x, y is reversed; we will thus drop variable subscripts since it suffices to consider any variable order.

If $\mathcal{F} = \{f_0, f_1, f_2\}$ has a common zero, it is also a zero of $\theta(f_0, f_1, f_2)$ for any value of new variables $\overline{x}$ and $\overline{y}$. Thus,

$$\Theta \times X = 0, \qquad (2)$$

whenever x, y are replaced by a common zero of f_0, f_1, f_2.

For polynomials $\{f_0, f_1, f_2\}$ to have a common zero, the equation (2) must be satisfied. If Θ is square and nonsingular, then its determinant must vanish, implying that under certain conditions, Θ is a resultant matrix. Even though this matrix is quite different from matrices constructed using the Sylvester dialytic method, there is a direct connection between the two which will be discussed later (see also [CK00b] and [CK02b]).

We are interested in identifying conditions when the resultant matrix Θ is **exact**, i.e., its determinant is exactly (up to a constant factor) the resultant. Also, when it is not, we are interested in predicting the extraneous factor in the determinant of Θ (at the very least, the degree of the extraneous factor).

Resultant is identified from a *projection operator*, a polynomial which is a determinant of some maximal minor of Θ. Since Θ is assumed to be a resultant matrix (see [KS96] and [BEM00]), it follows that

$$|X| \geq \max\left(\mu(\mathcal{A}_0, \mathcal{A}_1), \mu(\mathcal{A}_0, \mathcal{A}_2), \mu(\mathcal{A}_1, \mathcal{A}_2)\right),$$

and in unmixed case, where $\mathcal{A} = \mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2$,

$$|X| \geq 2\ \mathsf{Vol}(\mathcal{A}).$$

We are thus interested in analyzing the size and structure of the monomial set X; its size tells the number of columns in Θ and hence, whether or not, Θ is exact, which is the case when $|X| = 2\ \mathsf{Vol}(\mathcal{A})$.

3.1 The Dixon Polynomial and its Support

By $\sigma \Subset \langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle$, we mean $\langle \alpha, \beta, \gamma \rangle$ such that $\alpha \in \mathcal{A}_0$, $\beta \in \mathcal{A}_1$ and $\gamma \in \mathcal{A}_2$.

The Dixon polynomial above can be expressed using the Cauchy-Binet formula as a sum of Dixon matrices of 3-point set supports as shown below, (also see [CK02b] for a complete derivation).

$$\theta(f_0, f_1, f_2) = \sum_{\sigma \Subset \langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle} \sigma(\mathbf{c})\, \sigma(\mathbf{x}), \tag{3}$$

where

$$\sigma(\mathbf{c}) = \begin{vmatrix} a_\alpha & a_\beta & a_\gamma \\ b_\alpha & b_\beta & b_\gamma \\ c_\alpha & c_\beta & c_\gamma \end{vmatrix} \qquad \text{and} \qquad \sigma(\mathbf{x}) = \frac{1}{(\overline{x} - x)(\overline{y} - y)} \begin{vmatrix} x^{\alpha_x} y^{\alpha_y} & \overline{x}^{\alpha_x} y^{\alpha_y} & \overline{x}^{\alpha_x} \overline{y}^{\alpha_y} \\ x^{\beta_x} y^{\beta_y} & \overline{x}^{\beta_x} y^{\beta_y} & \overline{x}^{\beta_x} \overline{y}^{\beta_y} \\ x^{\gamma_x} y^{\gamma_y} & \overline{x}^{\gamma_x} y^{\gamma_y} & \overline{x}^{\gamma_x} \overline{y}^{\gamma_y} \end{vmatrix}.$$

In a generic case, where $\sigma(\mathbf{c})$ is not 0, the support of the Dixon polynomial is the union of supports of $\sigma(\mathbf{x})$ in the variables $\mathbf{x}$, where $\sigma(\mathbf{x})$ is the Dixon polynomial of the monomials corresponding to $\sigma = \langle \alpha, \beta, \gamma \rangle$.

Let

$$\Delta_{\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle} = \{\, \alpha \mid \mathbf{x}^\alpha \in \theta(f_0, f_1, f_2) \,\}.^2$$

Hence, in the generic case,

$$\Delta_{\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle} = \bigcup_{\sigma \Subset \langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle} \Delta_\sigma \qquad \text{where} \quad \Delta_\sigma = \{\, \alpha \mid \mathbf{x}^\alpha \in \sigma(\mathbf{x}) \,\}.$$

As seen from the above formula, in the generic case, the support of the Dixon polynomial as well as the size of the Dixon matrix are completely determined by the support of the polynomial system $\mathcal{F}$.

3.2 Unmixed systems

The emphasis of first part of this article is on unmixed polynomial systems, so we will try to simplify the notation a bit. In the unmixed case, since $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2$, we will drop the subscript and let $\mathcal{A}$ (where $\mathcal{A} = \mathcal{A}_0$) stand for the support of unmixed polynomial system, in which case $\Delta_{\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle} = \Delta_{\langle \mathcal{A}, \mathcal{A}, \mathcal{A} \rangle} = \Delta_{\mathcal{A}}$.

The following proposition shows that the translation of the support of polynomials in an unmixed system has no effect on the size of the support of the Dixon polynomial (and hence the size of the Dixon matrix).

Proposition 3.1 *Given an unmixed polynomial system with support $\mathcal{A}$, let $q_x = \min_{\alpha \in \mathcal{A}} \alpha_x$ and $q_y = \min_{\alpha \in \mathcal{A}} \alpha_y$.*

$$\Delta_{\mathcal{A}} = (0, 2\, q_y) + \Delta_{\mathcal{A} - q},^3$$

that is $\Delta_{\mathcal{A}}$ is a "shift" of the support of the Dixon polynomial of the support situated at the origin.

Proof: Since $\mathcal{A}$ is the support of polynomials $\{f_0, f_1, f_2\}$, it follows that

$$f_0 = x^{q_x} y^{q_y} g_0, \quad f_1 = x^{q_x} y^{q_y} g_1, \quad \text{and} \quad f_2 = x^{q_x} y^{q_y} g_2,$$

where $\mathcal{A} - q$ is the support of $\{g_0, g_1, g_2\}$. Therefore

$$\theta(f_0, f_1, f_2) = x^{q_x} y^{2\, q_y}\, \overline{x}^{2\, q_x} \overline{y}^{q_y}\, \theta(g_0, g_1, g_2),$$

by factoring monomials from the rows of the matrix in the expression for the Dixon polynomial (1). Hence the statement. □

Throughout the paper, in the unmixed case, it will be assumed without any loss of generality that $\mathcal{A}$ is situated at the origin, that is $\min_{\alpha \in \mathcal{A}} \alpha_x = 0$ and $\min_{\alpha \in \mathcal{A}} \alpha_y = 0$.

[2] By an abuse of notation, by $\mathbf{x}^\alpha \in \theta$, we mean that the monomomial $\mathbf{x}$ appears in polynomial θ with a non-zero coefficient.

[3] "$-$" is the regular vector subtraction.

4 Structure of the Dixon polynomial

This section analyzes the relationship between the support $\Delta_{\mathcal{A}}$ of the Dixon polynmial with the support $\mathcal{A}$ for generic unmixed polynomial systems. We first study the relation between Δ_σ and a simplex σ. We introduce the concept of the *support hull* of a support based on Manhattan distance. The notion of *enclosure* of a point is introduced. It is shown that $\Delta_{\mathcal{A}}$ is "enclosed" by the *projection sum* of $\mathcal{A}$. *Support complement* of a support with respect to its *bounding box* (which is the support of the associated bidegree polynomial system) is defined. The support complement can be used to give a complete description of $\Delta_{\mathcal{A}}$ in terms of the support of the Dixon polynomial corresponding to the associated bidegree system and the support complement.

4.1 Support Hull

Given two points on a line, one can describe a relationship between them as one being before the other with respect to some direction. We extend this notion to two dimensions; the Euclidean plane is split into quadrants. This way a point can be defined to be in some quadrant of the other point, similar to a point on a line is on one or the other side of the other point.

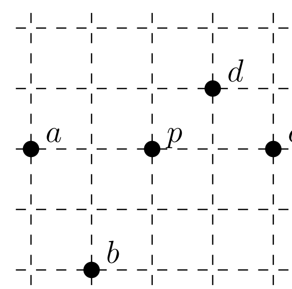

Figure 1: Example of $p \underset{k}{<} d$, when $k = (1,1)$.

Definition 4.1 *Given two points p and q in $\mathbb{N}^2$, and $k \in \mathbb{Z}_2^2$, where $k = \langle k_1, k_2 \rangle$,*

$$p \underset{k}{<} q \quad \Longleftrightarrow \quad \begin{cases} p_i < q_i & \textit{if } k_i = 1 \\ p_i \geq q_i & \textit{if } k_i = 0 \end{cases} \quad \textit{for } i = 1, 2,$$

and $p \underset{k}{\leqq} q$ whenever equality permitted for $k_i = 1$.

For example in figure 1, $p \underset{00}{<} b$ and also $p \underset{00}{<} a$, but not $p \underset{01}{\not<} a$. Also $b \underset{11}{<} p \underset{11}{<} d$, where $p \underset{10}{<} c$. In general $\underset{k}{<}$ is transitive, but it does not define a total order.

Similar to the concept of a convex hull of a support, we introduce the *support hull* of a support defined using the Manhattan distance.

Definition 4.2 *Given a support $\mathcal{P}$, a point p is in the support hull of $\mathcal{P}$, denoted by $p \trianglelefteq \mathcal{P}$, iff there exist points in $\mathcal{P}$ in every quadrant of p such that*

$$p \trianglelefteq \mathcal{P} \quad \Longleftrightarrow \quad \forall\, k \in \mathbb{Z}_2^2,\ \exists\, q \in \mathcal{P} \quad \textit{s.t.} \quad p \underset{k}{\leqq} q.$$

Two support hulls $\mathcal{P}$ and $\mathcal{Q}$ are equivalent if and only if for every p, $p \trianglelefteq \mathcal{P}$ iff $p \trianglelefteq \mathcal{Q}$.

Definition 4.3 *Given a support $\mathcal{P}$, a point $p \in \mathbb{N}^2$ is a* ***support interior*** *point of $\mathcal{P}$ if and only if*

$$\forall\, k \in \mathbb{Z}_2^2,\ \exists\, q \in \mathcal{P}, \quad \textit{where} \quad q \neq p, \quad \textit{s.t.} \quad p \underset{k}{\leqq} q.$$

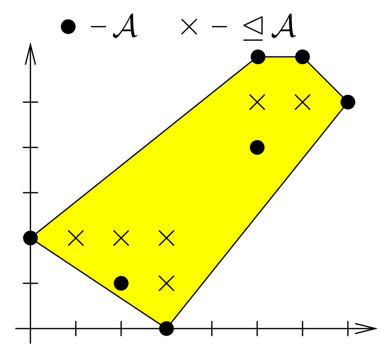

Figure 2: Points of support hull of $\mathcal{A}$.

In figure 2, all points shown belong to the support hull of $\mathcal{A}$. As can be seen from the figure, points of the support hull belong to the convex hull. This is true in general.

Proposition 4.1 *Given a point $p \in \mathbb{N}^2$ and a support $\mathcal{P} \subset \mathbb{N}^2$ then*

$$p \trianglelefteq \mathcal{P} \quad \Longrightarrow \quad p \in \mathsf{cHull}(\mathcal{P}).$$

Proof: Since $p \trianglelefteq \mathcal{P}$, it follows that there exists four points $\{q^{00}, q^{01}, q^{10}, q^{11}\} \subseteq \mathcal{P}$ such that $p \trianglelefteq \{q^{00}, q^{01}, q^{10}, q^{11}\}$. Since also $\{q^{00}, q^{01}, q^{10}, q^{11}\} \subset \mathsf{cHull}(\mathcal{P})$, line $[q^{00}, q^{10}]$ as well as line $[q^{01}, q^{11}]$ are part of convex hull of $\mathcal{P}$.

Since line $x = p_x$ intersects both lines at points s and t respectively, and since $q_y^{01}, q_y^{11} \geq p_y$ and $q_y^{00}, q_y^{10} \leq p_y$, it follows that segment $[s, t]$ on line $x = p_x$ containing p, is also a part of the convex hull of $\mathcal{P}$. $\square$

Intuitively, the notion of the convex hull of a support is based on the shortest Euclidean distance, whereas the notion of its support hull is based on the Manhatan distance.

Definition 4.4 *Given a support $\mathcal{P}$, a point $p \in \mathbb{N}^d$ is **enclosed** by the support hull of $\mathcal{P}$, denoted by $p \lhd \mathcal{P}$, if and only if*

$$p \lhd \mathcal{P} \qquad \Longleftrightarrow \qquad \forall\, k \in \mathbb{Z}_2^2,\ \exists\, q \in \mathcal{P} \quad s.t. \quad p \underset{k}{<} q.$$

Below, we will use these concepts to show that every point of $\Delta_{\mathcal{A}}$ is enclosed by some support.

4.2 Projection sum and its interior

First, we consider supports of size 3, called *simplexes*, as the support of the Dixon polynomial is the union of the supports of the Dixon polynomials of these simplexes.

Note that Δ_σ is the support of

$$\begin{aligned} \sigma(\mathbf{x}) &= \frac{1}{(\overline{x}-x)\,(\overline{y}-y)} \begin{vmatrix} x^{\alpha_x} y^{\alpha_y} & x^{\beta_x} y^{\beta_y} & x^{\gamma_x} y^{\gamma_y} \\ \overline{x}^{\alpha_x} y^{\alpha_y} & \overline{x}^{\beta_x} y^{\beta_y} & \overline{x}^{\gamma_x} y^{\gamma_y} \\ \overline{x}^{\alpha_x} \overline{y}^{\alpha_y} & \overline{x}^{\beta_x} \overline{y}^{\beta_y} & \overline{x}^{\gamma_x} \overline{y}^{\gamma_y} \end{vmatrix} \qquad (4) \\ &= y^{\alpha_y} \overline{x}^{\gamma_x} \frac{(x^{\alpha_x}\overline{x}^{\beta_x} - x^{\beta_x}\overline{x}^{\alpha_x})}{(\overline{x}-x)} \frac{(y^{\beta_y}\overline{y}^{\gamma_y} - y^{\gamma_y}\overline{y}^{\beta_y})}{(\overline{y}-y)} - \overline{x}^{\alpha_x} y^{\gamma_y} \frac{(x^{\beta_x}\overline{x}^{\gamma_x} - x^{\gamma_x}\overline{x}^{\beta_x})}{(\overline{x}-x)} \frac{(y^{\alpha_y}\overline{y}^{\beta_y} - y^{\beta_y}\overline{y}^{\alpha_y})}{(\overline{y}-y)}. \end{aligned}$$

We define below $\mathcal{D}_\sigma$ to stand for the support of $(\overline{x}-x)\,(\overline{y}-y)\sigma(\mathbf{x})$.

Given a simplex $\sigma = \langle \alpha, \beta, \gamma \rangle$, consider the following multi set (denoted using $\{\{$ and $\}\}$ to distinguish it from a set),

$$\mathcal{S}_{\mathcal{D}_\sigma} = \{\{+(\alpha_x, \alpha_y+\beta_y), -(\alpha_x, \alpha_y+\gamma_y), -(\beta_x, \alpha_y+\beta_y), +(\beta_x, \beta_y+\gamma_y), +(\gamma_x, \alpha_y+\gamma_y), -(\gamma_x, \beta_y+\gamma_y)\}\}.$$

From this multiset, $\mathcal{D}_\sigma$ is defined as a set of tuples as follows.

Definition 4.5 *Given a simplex $\sigma = \langle \alpha, \beta, \gamma \rangle$,*

$$\mathcal{D}_\sigma = \{\, p \mid +p \in \mathcal{S}_{\mathcal{D}_\sigma} \ or\ -p \in \mathcal{S}_{\mathcal{D}_\sigma},\ and\ \mathtt{multiplicity}(+p, \mathcal{S}_{\mathcal{D}_\sigma}) \neq \mathtt{multiplicity}(-p, \mathcal{S}_{\mathcal{D}_\sigma})\}.$$

Typically, for a generic σ, terms in $(\overline{x}-x)\,(\overline{y}-y)\sigma(\mathbf{x})$ do not cancel out; thus, $\mathcal{S}_{\mathcal{D}_\sigma}$ has unique occurrences of tuples. However, in some cases, e.g. $\sigma = \langle (2,0), (0,1), (2,1) \rangle$, positive and negative terms cancel out, as then

$$\mathcal{D}_\sigma = \{(0,1), (2,1), (0,2), (2,2)\}.$$

In general we let

$$\mathcal{D}_{\mathcal{A}} = \bigcup_{\sigma \in \mathcal{A}} \mathcal{D}_\sigma.$$

It is easy to describe points enclosed by $\mathcal{D}_\sigma$.

Proposition 4.2 *Given a simplex $\sigma = \langle \alpha, \beta, \gamma \rangle$, assume $\alpha_x \leq \beta_x \leq \gamma_x$. A point $p \in \mathbb{N}^d$ is enclosed by $\mathcal{D}_\sigma$, that is, $p \lhd \mathcal{D}_\sigma$ if and only if*

$$p \lhd \mathcal{D}_\sigma \qquad \Longleftrightarrow \qquad \begin{array}{c} \alpha_x \leq p_x < \beta_x \quad and \quad \alpha_y + \min(\beta_y, \gamma_y) \leq p_y < \alpha_y + \max(\beta_y, \gamma_y) \\ or \\ \beta_x \leq p_x < \gamma_x \quad and \quad \gamma_y + \min(\alpha_y, \beta) \leq p_y < \gamma_y + \max(\alpha_y, \beta_y). \end{array}$$

Proof: $\alpha_x \leq p_x < \gamma_x$ as otherwise clearly $p \not\lhd \mathcal{D}_\sigma$.

Case (i): $\alpha_x \leq p_x < \beta_x$: There are only two points in $\mathcal{D}_\sigma$ whose x coordinate is smaller than p_x:

$$(\alpha_x, \alpha_y + \beta_y) \qquad \text{and} \qquad (\alpha_x, \alpha_y + \gamma_y);$$

therefore, $p \lhd \mathcal{D}_\sigma$ if and only if $\alpha_y + \min(\beta_y, \gamma_y) \leq p_y < \alpha_y + \max(\beta_y, \gamma_y)$.

Case (ii): $\beta_x \leq p_x < \gamma_x$: There are only two points in $\mathcal{D}_\sigma$ whose x coordinate is bigger than p_x:

$$(\gamma_x, \alpha_y + \gamma_y) \qquad \text{and} \qquad (\gamma_x, \beta_y + \gamma_y);$$

therefore, $p \lhd \mathcal{D}_\sigma$ if and only if $\gamma_y + \min(\alpha_y, \beta) \leq p_y < \gamma_y + \max(\alpha_y, \beta_y)$. □

4.3 Support of the Dixon polynomial is Enclosed by its Projection Sum

First we will show that the support of the Dixon polynomial is enclosed by the projection sum of 3 points, which will enable us to show the result in general.

Theorem 4.1 *A point p belongs to the support of the Dixon polynomial of a simplex $\sigma = \{\alpha, \beta, \gamma\}$ if and only if it is enclosed by its projection sum $\mathcal{D}_\sigma$, that is*

$$p \in \Delta_\sigma \qquad \Longleftrightarrow \qquad p \lhd \mathcal{D}_\sigma.$$

Proof: W.l.o.g. assume $\alpha_x \leq \beta_x \leq \gamma_x$, then it can be seen from equation (5) that points of Δ_σ belong to one of the disjoint blocks

$$\left\{\, p \mid \alpha_x \leq p_x < \beta_x \quad \text{and} \quad \alpha_y + \min(\beta_y + \gamma_y) \leq p_y < \alpha_y + \max(\beta_y + \gamma_y) \,\right\},$$

or

$$\left\{\, p \mid \beta_x \leq p_x < \gamma_x \quad \text{and} \quad \gamma_y + \min(\alpha_y + \beta_y) \leq p_y < \max(\alpha_y + \beta_y) \,\right\}.$$

which is precisely a condition for $p \lhd \mathcal{D}_\sigma$ by Proposition 4.2. □

Theorem 4.2 *If the support $\mathcal{A}$ of a polynomial system is unmixed, then*

$$p \in \Delta_{\mathcal{A}} \qquad \Longleftrightarrow \qquad p \lhd \mathcal{D}_{\mathcal{A}}.$$

Proof: It will be shown that $p \lhd \mathcal{D}_{\mathcal{A}} \Longleftrightarrow p \lhd \mathcal{D}_\sigma$ for some $\sigma \Subset \mathcal{A}$, in which case

$$p \in \Delta_{\mathcal{A}} \quad \overset{\text{def}}{\Longleftrightarrow} \quad p \in \Delta_\sigma \quad \overset{\text{theorem 4.1}}{\Longleftrightarrow} \quad p \lhd \mathcal{D}_\sigma \quad \Longleftrightarrow \quad p \lhd \mathcal{D}_{\mathcal{A}}.$$

If $p \lhd \mathcal{D}_\sigma$, then $p \lhd \mathcal{D}_{\mathcal{A}}$, since by definition, $\mathcal{D}_\sigma \subseteq \mathcal{D}_{\mathcal{A}}$. To show that $p \lhd \mathcal{D}_{\mathcal{A}} \Longrightarrow p \lhd \mathcal{D}_\sigma$, for some $\sigma \Subset \mathcal{A}$, assume that $p \lhd \mathcal{D}_{\mathcal{A}}$, then for some

$$\{q^{00}, q^{01}, q^{10}, q^{11}\} \subseteq \mathcal{D}_{\mathcal{A}}, \qquad \text{we have} \qquad p \lhd \{q^{00}, q^{01}, q^{10}, q^{11}\},$$

where for $k = (i, j)$, $p \underset{k}{<} q^{ij}$. In general, by the definition of $\mathcal{D}_{\mathcal{A}}$,

$$q^{ij} = (\alpha^{ij}_x, \alpha^{ij}_y + \beta^{ij}_y) \qquad \text{for some } \alpha^{ij}, \beta^{ij} \in \mathcal{A} \quad \text{and} \quad i, j \in \{0, 1\}.$$

So for all $i, j \in \{0, 1\}$, there are 8 points α^{ij}, β^{ij} (not necessarily distinct) in $\mathcal{A}$ so that $p \lhd \mathcal{D}_{\mathcal{A}}$. We need to show that actually only 3 distinct points are needed. Since $p \lhd \{q^{00}, q^{01}, q^{10}, q^{11}\}$, the above 8 points satisfy the following four conditions

$$\begin{aligned}
\alpha^{00}_x &\leq p_x < \alpha^{10}_x, \\
\alpha^{01}_x &\leq p_x < \alpha^{11}_x, \\
\alpha^{00}_y + \beta^{00}_y &\leq p_y < \alpha^{01}_y + \beta^{01}_y, \\
\alpha^{10}_y + \beta^{10}_y &\leq p_y < \alpha^{11}_y + \beta^{11}_y.
\end{aligned}$$

To get three distinct points to form $\sigma \Subset \mathcal{A}$ we choose two of the three to be $\{\, \alpha^{00}, \alpha^{11}\}$. The third point is chosen on following case analysis.

Case (i): If $\alpha^{00}_y + \alpha^{11}_y \le p_y$, then consider set $\sigma = \{\alpha^{00}, \beta^{11}, \alpha^{11}\} \Subset \mathcal{A}$ and note that $p \lhd \mathcal{D}_\sigma$ since

$$\alpha^{00}_x \;\le\; p_x \;<\; \alpha^{11}_x \qquad \text{and} \qquad \alpha^{00}_y + \alpha^{11}_y \;\le\; p_y \;<\; \alpha^{11}_y + \beta^{11}_y .$$

Case (ii): If $p_y < \alpha^{00}_y + \alpha^{11}_y$, then consider set $\sigma = \{\alpha^{00}, \beta^{00}, \alpha^{11}\} \Subset \mathcal{A}$ and note that $p \lhd \mathcal{D}_\sigma$ since

$$\alpha^{00}_x \;\le\; p_x \;<\; \alpha^{11}_x \qquad \text{and} \qquad \alpha^{00}_y + \beta^{00}_y \;\le\; p_y \;<\; \alpha^{00}_y + \alpha^{11}_y .$$

Therefore, $p \lhd \mathcal{D}_\mathcal{A}$ implies that $p \lhd \mathcal{D}_\sigma$, and the statement of the theorem follows. □

$\mathcal{D}_\mathcal{A}$ is much easier to analyze than $\Delta_\mathcal{A}$. Since $\Delta_\mathcal{A}$ can be readily obtained from $\mathcal{D}_\mathcal{A}$, the set $\mathcal{D}_\mathcal{A}$ will be used in the proofs.

4.4 Support Complement

Definition 4.6 *Given an unmixed support $\mathcal{A}$ of a polynomial system $\mathcal{F}$, let $b = (b_x, b_y)$ where $b_x = \max_{\alpha \in \mathcal{A}} \alpha_x$ and $b_y = \max_{\alpha \in \mathcal{A}} \alpha_y$. Define the bounding box $\mathcal{B}$ of $\mathcal{A}$ to be the set*

$$\mathcal{B} = \{\; p = (p_x, p_y) \;\;|\;\; 0 \le p_x \le b_x \quad \textit{and} \quad 0 \le p_y \le b_y \,\} .$$

An unmixed polynomial system with support $\mathcal{B}$ is called a **bi-degree** system.

Dixon in [Dix08] generalized the Bezout method for full bi-degree polynomial systems, and established that matrices constructed using that method are exact, i.e., their determinants are the resultants of the polynomial systems. The support structure of the Dixon polynomial has been known, and is generalized to the n-degree systems in [KS96] and [Sax97].

Proposition 4.3 *The support of the Dixon polynomial of a polynomial system with support $\mathcal{B}$ is*

$$\Delta_\mathcal{B} = \{\; p = (p_x, p_y) \;\;|\;\; 0 \le p_x \le b_x - 1 \quad \textit{and} \quad 0 \le p_y \le 2\, b_y - 1 \,\} \qquad \textit{and hence} \quad |\Delta_\mathcal{B}| = 2\, b_x\, b_y .$$

Proof: Note that points $\{(0,0), (b_x, 0), (0, 2\, b_y), (b_x, 2\, b_y)\}$ are in $\mathcal{D}_\mathcal{B}$. Since $p \lhd \mathcal{D}_\mathcal{B}$ if and only if it is in the set stated by proposition, and since by theorem 4.2, $p \in \Delta_\mathcal{B} \Longleftrightarrow p \lhd \mathcal{D}_\mathcal{B}$ the proposition follows. □

An important point about box supports is that points in the support interior of the box support do not play any role in determining the support of the Dixon polynomial (which can be seen from the proof of the above Proposition 4.3); see also [KS97]. Later, we will give a precise description of points which do not influence the support of the Dixon polynomial. Identifying such points and not using them in computations can reduce the cost of algorithms based on this method.

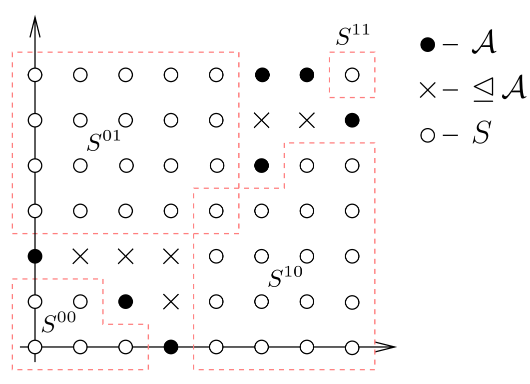

Figure 3: Support Complement

Definition 4.7 *Given an unmixed support $\mathcal{A}$ of a polynomial system, let*

$$S^k = \{\, s \;|\; s \in \mathcal{B} \quad \textit{and for all } \alpha \in \mathcal{A},\; s \not\le_k \alpha \,\} \quad \textit{for } k \in \mathbb{Z}^2,$$

$$\textit{and} \quad S = \bigcup_{k \in \mathbb{Z}^2_2} S^k .$$

See Figure 3 for the example of sets S^k. Note that S^k's are not necessarily disjoint as in the example, $S^{01} \cap S^{10} = \{(4, 3)\}$. The set S is called the *support complement* of $\mathcal{A}$ as justified by the following proposition.

Proposition 4.4 *Let $\mathcal{B}$ and S be the box support and support complement, respectively, of a support $\mathcal{A}$. A point p in $\mathcal{B}$ but not in S is support interior of $\mathcal{A}$, that is*

$$p \in \mathcal{B} - S \qquad \Longleftrightarrow \qquad p \trianglelefteq \mathcal{A}.$$

Proof: $p \in \mathcal{B} - S$ if and only if $p \notin S$, which happens if and only if for all $k \in \mathbb{Z}_2^2$, there exists $\alpha \in \mathcal{A}$ such that $p \underset{k}{\leq} \alpha$; hence, by Definition 4.3, $p \in \mathcal{B} - S$ if and only if $p \trianglelefteq \mathcal{A}$. □

One useful observation is that if $s = (s_x, s_y) \in S^k$, where $k = (k_1, k_2) \in \mathbb{Z}_2^2$ then

$$\begin{cases} s_x < b_x & \text{if } k_1 = 0, \\ s_x > 0 & \text{if } k_1 = 1, \end{cases} \quad \text{and} \quad \begin{cases} s_y < b_y & \text{if } k_2 = 0, \\ s_y > 0 & \text{if } k_2 = 1. \end{cases} \tag{5}$$

Also note that if $s \in S^k$, then for all $p \in \mathcal{B}$ such that $s \underset{k}{<} p$, $p \in S^k$.

4.5 Support of the Dixon polynomial through Support Complement

An interesting property of the support of the Dixon polynomial is that it admits a concise geometric description, given that it is a union of the supports of the Dixon polynomial of polynomial systems with smaller support sets. The following theorem gives the support of the Dixon polynomial in terms of how different the support of the polynomial system is from the bi-degree support. It also enables one to compute the support of the Dixon polynomial without expanding all determinants in the formula for the Dixon polynomial.

We define a set based on the support complement. This set is the "missing" part from the support $\Delta_{\mathcal{B}}$ of Dixon polynomial of the box support $\mathcal{B}$. Relating $\Delta_{\mathcal{B}}$ and $\Delta_{\mathcal{A}}$ in terms of difference between $\mathcal{B}$ and $\mathcal{A}$ yields precise description of the structure of the Dixon polynomial of polynomial system with support $\mathcal{A}$.

Definition 4.8 *For $k \in \mathbb{Z}_2^2$, let*

$$T^k = r^k + S^k \quad \textit{and} \quad T = \bigcup_{k \in \mathbb{Z}_2^2} T^k,$$

$$\textit{where} \quad r_x^k = -k_1 \quad \textit{and} \quad r_y^k = k_2(b_y - 1).$$

All points not in this set T but in $\Delta_{\mathcal{B}}$ are part of $\Delta_{\mathcal{A}}$.

Theorem 4.3 *The support of the Dixon polynomial of an unmixed polynomial system with support $\mathcal{A}$ is defined by the support complement of $\mathcal{A}$, that is*

$$\Delta_{\mathcal{A}} = \Delta_{\mathcal{B}} - T,$$

where T is defined in Definition 4.8.

The same theorem is independently proved in [Chi01]; the proof method seems to be quite different, however.

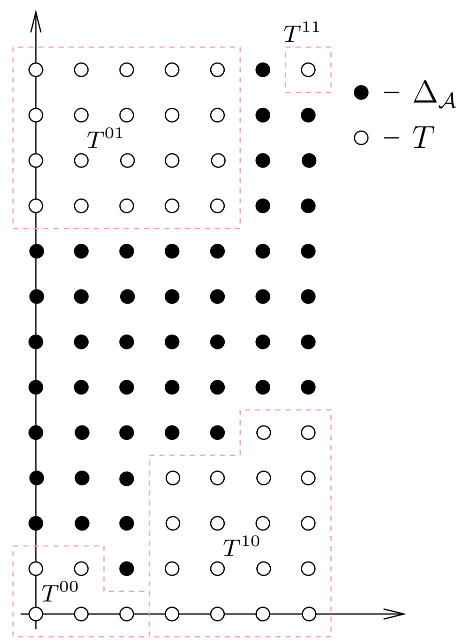

Figure 4: $\Delta_{\mathcal{A}}$ where $\mathcal{A}$ is from Figure 3

Proof: By Theorem 4.2, $p \lhd \mathcal{D}_\sigma \iff p \in \Delta_{\mathcal{A}}$, therefore we need to show that $p \in T \iff p \not\lhd \mathcal{D}_{\mathcal{A}}$. Since $T = \bigcup_{k \in \mathbb{Z}_2^2} T^k$, it is enough to show that for any $k \in \mathbb{Z}_2^2$, $p \in T^k \iff p \not\lhd \mathcal{D}_{\mathcal{A}}$. In particular, we will show that there is no $q \in \mathcal{D}_\sigma$ such that $p \underset{k}{<} q$, which will prove $p \in T^k \iff p \not\lhd \mathcal{D}_{\mathcal{A}}$.

We prove by contradiction, assuming the contrary that for some $k \in \mathbb{Z}_2^2$, there exists $q \in \mathcal{D}_\sigma$ such that $p \underset{k}{<} q$, then

$$q = (\alpha_x, \alpha_y + \beta_y),$$

for some $\alpha, \beta \in \mathcal{A}$. Since $p \in T^k$, it follows that $p = r^k + s$ for some $s \in S^k$. Since $p \underset{k}{<} q$, we have

$$\begin{cases} s_x \geq \alpha_x & \text{if } k_1 = 0, \\ s_x - 1 < \alpha_x & \text{if } k_1 = 1, \end{cases} \quad \text{and} \quad \begin{cases} s_y \geq \alpha_y + \beta_y & \text{if } k_2 = 0, \\ s_y + b_y - 1 < \alpha_y + \beta_y & \text{if } k_2 = 1. \end{cases} \tag{6}$$

Since $s \in S^k$, it follows that $s \not\leq_k \alpha$, in particular either (i) $s_x < \alpha_x$ when $k_1 = 0$ or (ii) $s_x \geq \alpha_x$ if $k_1 = 1$, or (iii) $s_y < \alpha_y$ when $k_2 = 0$ or (iv) $s_y \geq \alpha_y$ if $k_2 = 1$. But clearly all of these cases are incompatible with (6). Hence there is no such $q \in \mathcal{D}_{\mathcal{A}}$ such that $p \lessgtr_k q$ and hence $p \not\trianglelefteq \mathcal{D}_{\mathcal{A}}$. □

An immediate consequence of the above theorem is that the support interior points do not change the structure of the Dixon polynomial.

Corollary 4.3.1 *Given an unmixed support $\mathcal{A}$ of a polynomial system $\mathcal{F}$ and a point $p \in \mathbb{N}^2$, if p is a support interior of $\mathcal{A}$ then*

$$p \trianglelefteq \mathcal{P} \qquad \Longleftrightarrow \qquad \Delta_{\mathcal{A}} = \Delta_{\mathcal{A} \cup \{p\}}$$

that is, the presence of the monomial $x^{p_x} y^{p_y}$ in the polynomials of $\mathcal{F}$ does not effect the structure of the Dixon polynomial of $\mathcal{F}$.

Proof: By proposition 4.4 if $p \trianglelefteq \mathcal{A}$ then $p \notin S^k$ for any $k \in \mathbb{Z}_2^2$. $p \in \mathcal{A}$ or not, S^k's do not change and hence sets T^k are also invariant. Therefore by Theorem 4.3, presence of monomial $x^{p_x} y^{p_y}$ in $\mathcal{F}$ does not change structure of the Dixon polynomial of $\mathcal{F}$. □

4.6 Size of the Dixon Matrix

Proposition 4.5 $|T| = |S^{(0,0)}| + |S^{(1,0)}| + |S^{(0,1)}| + |S^{(1,1)}|$.

Proof: We only need to show that $T^k \cap T^l = \oslash$ for $k \neq l$ and $k, l \in \mathbb{Z}_2^2$, as $|T^k| = |S^k|$. Consider the opposite, that there exists $p \in T^k \cap T^l$, then by Definition 4.8,

$$r^k + s = p = r^l + t \qquad \text{for} \quad s \in S^k \text{ and } t \in S^l.$$

Since $k \neq l$, then either (i) $k_1 \neq l_1$ or (ii) $k_2 \neq l_2$.

Case (ii): W.l.o.g. assume $k_2 = 0$ and $l_2 = 1$; then $r^k = 0$ and $r^l = b_y - 1$ which implies $s_y = t_y + b_y - 1$. But since $s \in S^k$ and $t \in S^l$, by observation (5),

$$s_y < b_y \quad \text{and} \quad t_y > 0,$$

contradicting $s_y = t_y + b_y - 1$.

Case (i): w.l.o.g. assume that $k_1 = 0$ and $l_1 = 1$ and $k_2 = l_2$, then $s_x = t_x - 1$ and $s_y = t_y$. Since $s \in S^k$ and $t \in S^l$, there is no α in $\mathcal{A}$ such that $s \lessgtr_k \alpha$ or $t \lessgtr_l \alpha$, that is for all $\alpha \in \mathcal{A}$

$$s_x < \alpha_x \quad \text{or} \quad \begin{cases} s_y < \alpha_y & \text{if } k_2 = 0, \\ s_y > \alpha_y & \text{if } k_2 = 1, \end{cases} \qquad \text{and also} \qquad t_x > \alpha_x \quad \text{or} \quad \begin{cases} t_y < \alpha_y & \text{if } k_2 = 0, \\ t_y > \alpha_y & \text{if } k_2 = 1. \end{cases}$$

Since we have already established that $s_x = t_x - 1$ and $s_y = t_y$, this implies that for all $\alpha \in \mathcal{A}$, $\alpha_y < s_y$ when $k_2 = 1$ or $\alpha_y > s_y$ when $k_2 = 0$, which is impossible because $s \in \mathcal{B}$. □

We can now precisely express the size of the Dixon matrix of unmixed generic polynomial system with support $\mathcal{A}$.

Theorem 4.4 (Main) *The size of the support of the Dixon polynomial of an unmixed polynomial system $\mathcal{F}$ with support $\mathcal{A}$ is*

$$|\Delta_{\mathcal{A}}| = 2\, b_x\, b_y - |S^{00}| - |S^{01}| - |S^{10}| - |S^{11}|.$$

Proof: Since by Theorem 4.3, $\Delta_{\mathcal{A}} = \Delta_{\mathcal{B}} - T$; since by Proposition 4.3, $|\Delta_{\mathcal{B}}| = 2\, b_x\, b_y$ and $|T| = |S^{00}| + |S^{01}| + |S^{10}| + |S^{11}|$ by proposition 4.5. □

$\Delta_{\mathcal{A}}$ is dependent on the variable order used in $\theta_{\mathcal{A}}$, but the size of $\Delta_{\mathcal{A}}$ is the same for any variable order if $\mathcal{A}$ is unmixed. The number of columns is determined by the size of the support in terms of variables x, y. On the other hand, the number of rows is determined by the size of support in terms of variables $\overline{x}, \overline{y}$, which is the same as if the variable order is reversed and the support is considered in terms of variables x, y.

The above observation thus implies that the Dixon matrix is square for unmixed polynomial systems, but it need not be square for mixed polynomial systems. For example, for a polynomial system with support $\mathcal{A}_0 = \{(0,0), (1,1), (0,1)\}$, $\mathcal{A}_1 = \{(0,0), (1,0)\}$, and $\mathcal{A}_2 = \{(0,0), (1,0)\}$, its Dixon matrix is of size 2×1.

5 Exact Cases

In this section, we relate the size of the Dixon matrix associated with a given polynomial system, which is determined by the size of the support of its Dixon polynomial, to the BKK bound on the number of its toric roots, which is determined by the mixed volume of the Newton polytopes of its supports. We identify necessary and sufficient conditions on the support of the polynomial system under which the Dixon matrix is exact in the sense that its size is precisely the BKK bound. When these conditions on the support are not satisfied, we give an estimate on the degree of the extraneous factor in the projection operator extracted from the Dixon matrix by relating its size to the BKK bound.

How does the size of the Dixon matrix compare with the BKK bound of $\mathcal{F}$? For the unmixed case, if the size of the Dixon matrix equals the BKK bound of any two polynomials in $\mathcal{F}$, then the matrix is exact, i.e., its determinant is exactly the toric resultant. To see the relationship between the BKK bound which is defined in terms of Newton polytopes, and the size of the Dixon matrix, we can characterize how different the convex hull of the support $\mathcal{A}$ is from the box support using *corners*. Let

$$\mathcal{Q} = \mathcal{B}_{\mathcal{A}} - \mathsf{cHull}\,(\mathcal{A})\,,$$

which can be split into four disjoint, not necessarily convex, polyhedral sets. For $k = \langle k_1, k_2 \rangle \in \mathbb{Z}_2^2$, let $b^k = (k_1\, b_x, k_2\, b_y)$, and define

$$\mathcal{Q}^k = \{\, q \mid q \in \mathcal{Q} \text{ and the open sided segment } [b^k, q) \subset \mathcal{Q}\}.$$

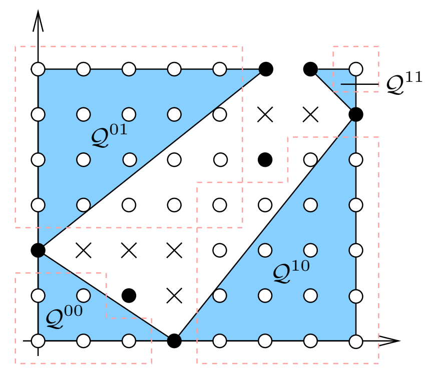

Figure 5: Newton polytope Complement

Figure 5 shows the Newton polytope complement for the earlier example shown in Figures 3 and 4.

For an unmixed polynomial system, the BKK bound, which is the mixed volume of any two polynomials with the support $\mathcal{A}$, is

$$\mu(\mathcal{A},\mathcal{A}) = 2\,\mathsf{Vol}(\mathcal{A}) = 2\mathsf{Vol}(\mathcal{B}_{\mathcal{A}}) - 2\,\mathsf{Vol}\big(\mathcal{B}_{\mathcal{A}} - \mathsf{cHull}\,(\mathcal{A})\big) = 2\,\mathsf{Vol}(\mathcal{B}_{\mathcal{A}}) - 2\,\mathsf{Vol}(\mathcal{Q}).$$

Since T^k's (see Definition 4.8 above) are disjoint, the Dixon matrix is exact if it can be proved that

$$\begin{aligned} 2\,\mathsf{Vol}(Q) &= 2\,\mathsf{Vol}(\mathcal{Q}^{00}) + 2\,\mathsf{Vol}(\mathcal{Q}^{01}) + 2\,\mathsf{Vol}(\mathcal{Q}^{10}) + 2\,\mathsf{Vol}(\mathcal{Q}^{11}) \\ &= |S^{00}| + |S^{01}| + |S^{10}| + |S^{11}|. \end{aligned}$$

In the proof of the following theorem, we need to look at the support hull.

Definition 5.1 *Given a support $\mathcal{P}$, let*

$$V_{\mathcal{P}} = \{\, \beta \in \mathcal{P} \mid \exists\, k \in \mathbb{Z}_2^2 \text{ s.t. for all } \alpha \in \mathcal{P},\ \alpha \neq \beta \implies \beta \not\leq_k \alpha \,\}.$$

$V_{\mathcal{P}}$ is called the support vertices of the support hull of $\mathcal{P}$.

Intuitively, support vertices are "extreme" points of the support; they have at least one empty quadrant. Further, the vertices of the convex hull of a given support are support hull vertices, but not all support vertices are the convex hull vertices. In Figure 2 earlier in Subsection 4.1, filled points are support hull vertices, and crossed points are support interior. As can be seen from the example, points $(2,1)$ and $(5,4)$ are in the support hull but they are not the vertices of the convex hull of $\mathcal{A}$.

Proposition 5.1 *Given the support complement S of a given support $\mathcal{A}$ and its Newton polytope complement $\mathcal{Q}$, the following two properties hold:*

(i) $|S^k| \leq 2\,\mathsf{Vol}(\mathcal{Q}^k)$ *and*

(ii) $|S^k| = 2\,\mathsf{Vol}(\mathcal{Q}^k)$ *if and only if each* S^k *is a rectangle.*

Proof: Let $V_{\mathcal{A}}$ be the support vertices in the support hull $\mathcal{A}$. This set can be partitioned into four subsets, based on the quadrant $k \in \mathbb{Z}_2^2$.

$$V_{\mathcal{A}}^k = \left\{ \beta \in \mathcal{A} \mid \text{for all } \alpha \in \mathcal{A}, \text{ where } \alpha \neq \beta, \text{ s.t. } \beta \not\leq_k \alpha \right\}. \tag{7}$$

Depending upon the value of $k = \langle k_0, k_1 \rangle$, if $k_1 = 0$, then $V_{\mathcal{A}}^k$ is sorted on x coordinate in the ascending order; if $k_1 = 1$, then sort $V_{\mathcal{A}}^k$ in the descending order. This will ensure that after sorting, $V_{\mathcal{A}}^k = [v_1, \ldots, v_n]$ has the property that $v_{i,x} < v_{i+1,x}$ if $k_1 = 0$ and $v_{i+1,x} < v_{i,x}$ if $k_1 = 1$. Also $v_{i,y} < v_{i+1,y}$ if $k = (1,0)$ or $k = (0,1)$ and $v_{i,y} > v_{i+1,y}$ otherwise; this follows from the properties of $V_{\mathcal{A}}^k$.

Let $[p, q]$ be a rectangular region in $\mathbb{N}^2$, where $\alpha \in [p, q]$ if and only if α_x is between p_x, q_x and α_y is between p_y, q_y. Split S^k into such rectangular regions $\{R_1, \ldots, R_{n-1}\}$ where $R_i = [p, q]$ for $p, q \in \mathbb{N}^d$, such that

$$p = (v_{i,x}, v_{i,y} + (-1)^{k_2+1}) \qquad \text{and} \qquad q = (v_{i+1,x} + (-1)^{k_2}, k_2 b_y).$$

Each region is disjoint and their union covers the entire S^k, that is,

$$S^k = \bigcup_{i=1}^{n-1} R_i \qquad \text{and} \qquad R_i \cap R_j = \oslash \qquad \Longrightarrow \qquad |S^k| = \sum_{i=1}^{n-1} |R_i|.$$

For each rectangular region $R_i = [p, q]$, which is determined by the vertex points v_i and v_{i+1}, associate a triangle $\tau_i = \{v_i, v_i', v_{i+1}'\} \subset \mathbb{N}^2$, where $v_i' = (v_{i,x}, k_2 b_y)$ and $v_{i+1}' = (v_{i+1,x}, k_2 b_y)$. Note that

$$2\,\mathsf{Vol}(\tau_i) = |R_i|.$$

Below, it is proved that

$$\sum_{i=1}^{n-1} \mathsf{Vol}(\tau_i) \leq \mathsf{Vol}(Q^k), \tag{8}$$

from which the (ii) part of the statement, $|S^k| \leq 2\,\mathsf{Vol}(\mathcal{Q}^k)$, follows. Each side of the inequality (8) is calculated below.

Since the vertices of the convex hull of a given support $\mathcal{A}$ are also the vertices in its support hull, $\mathcal{Q}^k$ can be described using $V_{\mathcal{A}}^k = [v_1, \ldots, v_n]$. Let $H^k = [h_1, \ldots, h_m]$ where $m \leq n$, stand for the vertices in the convex hull of $\mathcal{A}$ in the k^{th} quadrant; for each $h_j = v_i$ and $h_{j+1} = v_l$, $i < l$, that is, the order of V^k is preserved. The volume of $\mathcal{Q}^k$, where $k = (k_1, k_2) \in \mathbb{Z}_2^2$, can be computed from H^k as:

$$2\,\mathsf{Vol}(Q^k) = \sum_{i=1}^{m-1} |h_{i+1,x} - h_{i,x}|\,|\,2\,k_2\,b_y - h_{i+1,y} - h_{i,y}|.$$

Let $[v_s, v_{s+1}, \ldots, v_{s+t}]$ be a sublist of V^k for some $s \in \{1, \ldots, n-1\}$. For some $0 < t \leq n - s$, such that $v_s, v_{s+t} \in H^k$ and $v_{s+i} \notin H^k$ for $0 < i < t$. Inequality (8) can be split into a sum over such sublists of V^k. It thus suffices to show that

$$\sum_{i=s}^{s+t-1} 2\,\mathsf{Vol}(\tau_i) \;\leq\; |v_{s+t,x} - v_{s,x}|\,|\,2\,k_2\,b_y - v_{s+t,y} - v_{s,y}|, \tag{9}$$

Since

$$2\,\mathsf{Vol}(\tau_i) = |v_{i+1,x} - v_{i,x}||k_2\,b_y - v_{i,y}|, \qquad \text{and} \qquad |v_{s+t,x} - v_{s,x}| = \sum_{i=s}^{s+t-1} |v_{i+1,x} - v_{i,x}|,$$

substituting them into (9), using the properties that $2\,k_2\,b_y - v_{s+t,y} - v_{s,y} \geq |k_2\,b_y - v_{i,y}|$ for any $s \leq i \leq s + t$, (9) is proved. Hence the proof of the part (ii) of the statement.

Note that inequality (9) will become equality if (a) $t = 1$ and (b) $v_{s+t,y} = 0$ if $k_2 = 0$ and $v_{s+t,y} = b_y$ otherwise; this is only the case for $n = 2$, i.e., there are only two support vertices implying that S^k is a rectangle.

On the other hand, if S^k is a rectangle, then $n = 2$; further, $v_{2,y} = 0$ if $k_2 = 0$ and $v_{2,y} = b_y$ otherwise. In that case, the inequality (8) becomes equality which implies that $|S^k| = 2\,\mathsf{Vol}(\mathcal{Q}^k)$. □

From the above proposition, there is a nice characterization of all bivariate unmixed polynomial systems for which the Dixon method computes the resultant exactly.

Theorem 5.1 *Given an unmixed generic polynomials system with support $\mathcal{A}$ such that $\{S^{00}, S^{01}, S^{10}, S^{11}\}$ is its support complement, the Dixon method computes its resultant exactly if and only if each S^k is a rectangle for $k \in \mathbb{Z}_2^2$.*

In contrast to the results in [Chi01], Theorem 5.1 thus provides a necessary and sufficient condition on the support of an unmixed generic bivariate polynomial system for which the Dixon method computes the resultant exactly. Furthermore, Theorem 6.1 below also gives an estimate of the degree of the extraneous factor in the projection operator computed by the Dixon method if an unmixed generic bivariate polynomial system does not satisfy this condition. These results are thus strict generalizations of the results in [Chi01].

Another implication of the above theorem together with Corollary 4.3.1 is that inclusion of terms corresponding to support-interior points in a polynomial system do not change the support of the Dixon polynomial and hence, the size of Dixon matrix and the degree of projection operator. However, inclusion of terms corresponding to points in the convex hull of the support but which are not support-interior, can contribute to the extraneous factors in the projection operator. But that is not the only source of extraneous factors in a projection operator. Even polynomial systems whose support does not have any points inside its convex hull can have extraneous factors in the projection operator computed by the Dixon method; consider example 5, for instance, in section 8 where examples are discussed.

6 Degree of Extraneous Factors

From the results of the previous section, we also have another key result of this paper.

Theorem 6.1 *The size of the Dixon matrix of an unmixed generic polynomial system $\mathcal{F} = \{f_0, f_1, f_2\}$ with a support $\mathcal{A}$ is*

$$|\Delta_{\mathcal{A}}| = \mu(\mathcal{A}, \mathcal{A}) + \sum_{k \in \mathbb{Z}_2^2} \left(2\mathsf{Vol}(\mathcal{Q}^k) - \left|S^k\right|\right) = \mu(\mathcal{A}, \mathcal{A}) + D_e.$$

And, D_e is an upper bound on the degree of the extraneous factor in the projection operator expressed in the coefficients of f_0, f_1 and f_2, and extracted from the Dixon matrix.

The proof of this theorem follows from Proposition 5.1 and the discussion immediately above Proposition 5.1 in the previous section.

In [CK00a], a method based on partitioning the support of an unmixed polynomial system is given for estimating the degree of the extraneous factor in the projection operator extracted from the associated Dixon matrix. The above theorem generalizes that result; instead of breaking up the support into smaller supports, it gives a better insight into the existence of extraneous factors. Further, the estimate on the degree of an extraneous factor can be calculated efficiently using the above relation.

6.1 Computing the degree of extraneous factor from $\mathcal{A}$

As discussed above, the degree of the extraneous factor in a projection operator is given by $|\Delta_{\mathcal{A}}| - 2\,\mathsf{Vol}(\mathcal{A})$. To estimate it, a method to compute $|\Delta_{\mathcal{A}}|$ and $\mathsf{Vol}(\mathcal{A})$ is needed. This amounts to computing $|S^{00}| + |S^{11}| + |S^{01}| + |S^{10}|$ and $\mathsf{Vol}(\mathcal{Q})$.

From the proof of Proposition 5.1, one way to calculate the size of S^k is to compute the support vertices of the support hull of $\mathcal{A}$ in the k^{th} quadrant. From these, $\mathsf{Vol}(\mathcal{Q}^k)$ can also be computed.

Given a set $\mathcal{A}$ and a quadrant $k \in \mathbb{Z}_2^2$, Algorithm 1 computes the set V^k. Function $\texttt{Sort}^k(\mathcal{A})$ sorts the elements of $\mathcal{A}$, first on the x coordinate and then on y coordinate, for those points with the same x coordinate. Depending on value of $k = \langle k_1, k_2 \rangle$, elements in $\mathcal{A}$ are sorted in the ascending order on x coordinate if $k_1 = 0$, otherwise if $k_1 = 1$, they are sorted in the descending order. For y, $k_2 = 0$, then sorting is done in the descending order, and in the ascending order otherwise.

The comparison function $\texttt{less}(i, a, b)$ returns true if $a < b$ when $i = 1$ or $a > b$ when $i = 0$ and false otherwise.

After sorting, the algorithm selects "extreme" k^{th} quadrant points out into a list. With the exception of sorting, all other steps are of linear complexity; hence, the total cost is dominated by the cost of sorting, and therefore the algorithm is of $O(n \log n)$, where $n = |\mathcal{A}|$.

Proposition 6.1 *Algorithm 1 computes $V_{\mathcal{A}}$, the support vertices of the support hull of a given support $\mathcal{A}$, as in Definition 5.1 and $V_{\mathcal{A}}^k$ in each quadrant as in (7) in the proof of Proposition 5.1.*

Proof: It is shown below that every point p returned by the algorithm is a support vertex in k^{th} quadrant; in other words, for all $q \in \mathcal{A}$, where $q \neq p$, $p \underset{k}{\not\lneqq} q$.

The proof is by contradiction. Assume that there exist a $q \in \mathcal{A}$, $q \neq p$ s.t. $p \underset{k}{\lneqq} q$; moreover w.l.o.g. assume that q is maximal, that is there is no other point $r \in \mathcal{A}$ such that $q \underset{k}{\lneqq} r$. Then

$$\begin{cases} p_x \geq q_x & \text{if } k_1 = 0, \\ p_x \leq q_x & \text{if } k_1 = 1, \end{cases} \quad \text{and} \quad \begin{cases} p_y \geq q_y & \text{if } k_2 = 0, \\ p_y \leq q_y & \text{if } k_2 = 1. \end{cases}$$

In the list $[\alpha_1, \ldots, \alpha_n]$ computed by $\texttt{Sort}^k(\mathcal{A})$, q will appear before p.

Since q is maximal, at some point statement $cur \leftarrow q$ will be reached; by the time $\alpha_i = p$,

$$\begin{cases} cur_x \geq q_x & \text{if } k_1 = 0, \\ cur_x \leq q_x & \text{if } k_1 = 1, \end{cases} \quad \text{and} \quad \begin{cases} cur_y \leq q_y \leq p_y & \text{if } k_2 = 0, \\ cur_y \geq q_y \geq p_y & \text{if } k_2 = 1, \end{cases}$$

and hence, p will not be added to the list, contradicting the assumption that p is returned by the algorithm.

It is now shown that the algorithm computes all such points, i.e., there does not exist any p in $\mathcal{A}$ such that p is a support vertex in k^{th} quadrant, but is not returned by the algorithm. The proof is again by contradiction. Suppose a support vertex $p \in \mathcal{A}$ is not returned by the algorithm. Then one of the two things happened: (i) it was never the case that $cur = p$, or (ii) for some $2 \leq i \leq n$, $\alpha_i = cur = p$ and $\alpha_{j,x} = p_x$ for $j = i, \ldots, n$.

Case (i): Let $p = \alpha_j$, for some $j \in \{2, \ldots, n\}$. Since $cur \neq p$, it must be the case that there exists $cur = \alpha_i$ for $i < j$, such that

$$\begin{cases} \alpha_{i,y} \leq \alpha_{j,y} & \text{if } k_2 = 0, \\ \alpha_{i,y} \geq \alpha_{j,y} & \text{if } k_2 = 1. \end{cases}$$

i.e., $p = \alpha_{j,y} \underset{k}{\lneqq} \alpha_{i,y}$ contradicting the assumption that p is a support vertex in k^{th} quadrant.

Algorithm 1: $\texttt{SupportVertices}(k, \mathcal{A})$

Data : Support $\mathcal{A}$ and quadrant $k = (k_1, k_2) \in \mathbb{Z}_2^2$.
Result : Support vertices V^k.
begin
  $[\alpha_1, \ldots, \alpha_n] \leftarrow \texttt{Sort}^k(\mathcal{A})$;
  $cur \leftarrow \alpha_1$;
  $V^k \leftarrow \{\}$;
  for $i = 2, \ldots, n$ **do**
    if $\texttt{less}(k_2, cur, \alpha)$ **then**
      if $cur_x <> \alpha_{i,x}$ **then**
        $V^k \leftarrow V^k$ $\texttt{append}$ cur;
      $cur \leftarrow \alpha_i$;
end

Case (ii): Since $[\alpha_1, \ldots, \alpha_d]$ are sorted with respect to the k^{th}-quadrant, it follows that p_x is either the maximum or the minimum x coordinate of $\mathcal{A}$. But since p was not added to the vertex list, this implies that there exists α_j, such that $\alpha_{j,x} = p_x$ and $\alpha_{j,y} < p_x$ if $k_2 = 0$ and $\alpha_{j,y} > p_x$ otherwise. In that case, $p \underset{k}{\lneqq} \alpha_j$, which means p is not a support vertex in k^{th} quadrant, contradicting the assumption.

Hence the Algorithm 1 computes precisely the set $V_{\mathcal{A}}^k$. □

After support vertices are computed, the size of the support complement can be computed as shown in Figure 6 using Algorithm 2, which is derived from the proof of proposition 5.1. Its complexity is dominated by `SupportVertices`$(k, \mathcal{A})$, which has the same complexity as sorting. Hence, the entire procedure of determining the size of the support complement and hence, the degree of the projection operator is of complexity $O(n \log n)$, where n is the size of the support $\mathcal{A}$.

Proposition 6.2 *Algorithm 2 computes* $|S^{00}| + |S^{01}| + |S^{10}| + |S^{11}|$ *of a given support* $\mathcal{A}$.

Algorithm 2: Compute complement size

Data : Support $\mathcal{A}$.

Result : s-number of points in support complement.

$s \leftarrow 0$;

$b_y \leftarrow \max_{a \in \mathcal{A}} a_y$;

foreach $k \in \mathbb{Z}_2^2$ **do**

  $[\alpha_1, \ldots, \alpha_n] \leftarrow$ `SupportVertices`$(k, \mathcal{A})$;

  for i *from 1 to* $n-1$ **do**

    $s \leftarrow s + |\alpha_{i+1,x} - \alpha_{i,x}|\,|k_2\, b_y - \alpha_{i,y}|$;

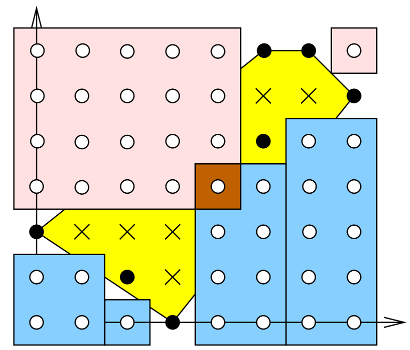

Figure 6: Computing $|S^{00}| + |S^{01}| + |S^{10}| + |S^{11}|$

Proof: The algorithm computes the size of S^k separately and then sums them up. It was shown in the proof of Proposition 5.1 that any $k \in \mathbb{Z}_2^2$,

$$|S^k| = \sum_{i=1}^{n-1} |\alpha_{i+1,x} - \alpha_{i,x}|\,|\,k_2\, b_y - \alpha_i, y|,$$

where $V^k = [\alpha_1, \ldots, \alpha_n]$ is the sorted list of support vertices computed by Algorithm 1. Further,

$$|S^k| = \sum_{i=1}^{n-1} 2\mathsf{Vol}(\tau_i);$$

the algorithm just computes this sum. □

Thus, for an unmixed bivariate polynomial system, it can be predicted exactly from the support, whether or not the Dixon method computes the resultant exactly, and if not, what the degree of the extraneous factor is in terms of the coefficients of one of the polynomials of the polynomial system.

7 Dixon Multiplier Matrix

As the reader would have noticed, the Dixon matrix above has, in general, complex entries; unlike in the Sylvester, Macaulay and sparse resultant formulations, where matrix entries are either zeros or coefficients of terms appearing in a polynomial system, entries in the Dixon matrix are determinants of the coefficients. For the bivariate case, entries are 3×3 determinants.

In [CK00b], we proposed a method for constructing Sylvester-type resultant matrices based on the Dixon formulation. Below, we review a generalization of that construction which has been recently developed; more details can be found in [CK02c]. We also show a relationship between these matrices and the Dixon matrices. The results in the previous section about the relationship between the support of the Dixon polynomial and the support of the polynomial system can be applied to the size of the Dixon multiplier matrices case as well.

Let $\mathcal{F}$ be a generic polynomial system $\{f_0, f_1, f_2\}$ with support $\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle$. Given an $\alpha \in \mathbb{N}^2$, the Dixon polynomial of $\mathcal{F}$ can be rewritten as

$$x^{\alpha_x} y^{\alpha_y}\, \theta(f_0, f_1, f_2) = f_0\, \theta(x^{\alpha_x} y^{\alpha_y}, f_1, f_2) + f_1\, \theta(f_0, x^{\alpha_x} y^{\alpha_y}, f_2) + f_2\, \theta(f_0, f_1, x^{\alpha_x} y^{\alpha_y}).$$

In section 3, the Dixon polynomial was expressed through the Dixon matrix as $\theta(f_0, f_1, f_2) = \overline{X} \Theta X$. Putting both expressions for the Dixon polynomial together, we get

$$\begin{aligned}
x^{\alpha_x} y^{\alpha_y}\, \theta(f_0, f_1, f_2) &= f_0\, \theta(x^{\alpha_x} y^{\alpha_y}, f_1, f_2) + f_1\, \theta(f_0, x^{\alpha_x} y^{\alpha_y}, f_2) + f_2\, \theta(f_0, f_1, x^{\alpha_x} y^{\alpha_y}) \\
&= f_0\, (\overline{X}_0\, \Theta_0\, X_0) + f_1\, (\overline{X}_1\, \Theta_1\, X_1) + f_2\, (\overline{X}_2\, \Theta_2\, X_2) \\
&= \overline{X}_0\, \Theta_0\, (X_0\, f_0) + \overline{X}_1\, \Theta_1\, (X_1\, f_1) + \overline{X}_2\, \Theta_2\, (X_2\, f_2) \\
&= \overline{Y}\, (\Theta_0 : \Theta_1 : \Theta_2) \begin{pmatrix} X_0 f_0 \\ X_1 f_1 \\ X_2 f_2 \end{pmatrix} \\
&= \overline{Y} \times (T \times (M_\alpha \times Y)),
\end{aligned}$$

where

$$T = (\Theta_0 : \Theta_1 : \Theta_2)\,, \qquad \overline{Y} = \overline{X}_0 \cup \overline{X}_1 \cup \overline{X}_2, \qquad \text{and} \qquad M_\alpha \times Y = \begin{pmatrix} X_0 f_0 \\ X_1 f_1 \\ X_2 f_2 \end{pmatrix}.$$

If $\mathcal{F} = \{f_0, f_1, f_2\}$ has a common solution, then $x^{\alpha_x} y^{\alpha_y}\, \theta(f_0, f_1, f_2) = 0$ and consequently,

$$\overline{Y} \times (T \times (M_\alpha \times Y)) = 0$$

for any values of $\overline{x}$ and $\overline{y}$. Hence, $T \times (M_\alpha \times Y) = 0$ whenever a solution of $\mathcal{F}$ is substituted into monomial vector Y. Because of the properties of the Dixon matrix and the fact that matrix T is too small to "contain" the resultant, the maximal minor of M_α is a projection operator. Consequently, M_α is a resultant matrix, henceforth called a *the Dixon Multiplier matrix*; see [CK02b] for more details.

The sets X_0, X_1 and X_2 of terms are the *multiplier sets* for f_0, f_1, f_2, respectively. They are monomials of the following Dixon polynomials, and their supports are expressed as follows:

$$\begin{aligned}
X_0 &= \{\, x^{p_x} y^{p_y} \mid x^{p_x} y^{p_y} \in \theta(x^{\alpha_x} y^{\alpha_y}, f_1, f_2)\}, \qquad & \mathcal{X}_0 = \Delta_{\langle \{\alpha\}, \mathcal{A}_1, \mathcal{A}_2 \rangle}, \\
X_1 &= \{\, x^{p_x} y^{p_y} \mid x^{p_x} y^{p_y} \in \theta(f_0, x^{\alpha_x} y^{\alpha_y}, f_2)\}, \qquad & \mathcal{X}_1 = \Delta_{\langle \mathcal{A}_0, \{\alpha\}, \mathcal{A}_2 \rangle}, \\
X_2 &= \{\, x^{p_x} y^{p_y} \mid x^{p_x} y^{p_y} \in \theta(f_0, f_1, x^{\alpha_x} y^{\alpha_y})\}, \qquad & \mathcal{X}_2 = \Delta_{\langle \mathcal{A}_0, \mathcal{A}_1, \{\alpha\} \rangle},
\end{aligned}$$

for some monomial $x^{\alpha_x} y^{\alpha_y}$ for $\alpha \in \mathbb{N}^2$.

It is shown in [CK02c] that for an unmixed polynomial system $\mathcal{F}$ with support $\mathcal{A}$, if $\alpha \trianglelefteq \mathcal{A}_0$ (see Definition 4.2), that is, α belongs to the support hull of $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2$, then

$$\Delta_{\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle} = \Delta_{\langle \{\alpha\}, \mathcal{A}_1, \mathcal{A}_2 \rangle}.$$

Hence,

$$\mathcal{X}_0 = \mathcal{X}_1 = \mathcal{X}_2 = \Delta_{\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle}.$$

In other words, the monomials of the Dixon polynomial and the multiplier sets remain the same.

It is proved in [CK00b] that for the special case of $\alpha = (0, 0)$, the matrix M_α is a Sylvester-type resultant matrix with entries 0 and coefficients of terms in polynomials in $\mathcal{F}$. Further, a projection operator can be extracted as the determinant of a rank submatrix of M_α [KSY94].[4] The matrix T is called the *transformation*

[4] In [CK00b], the monomial 1 is used for the construction of the Dixon multiplier matrices, which are called **sparse Dixon** matrices in [CK00b]. The above construction is a generalization of the construction in [CK00b]. This generalization turns out to be particularly useful for constructing "good" Dixon multiplier matrices for mixed polynomial systems; the determinants of such multiplier matrices have smaller degree extraneous factors in the associated projection operators.

matrix, and it relates the Dixon matrix to the associated Sylvester-type matrix (called the sparse Dixon matrix in [CK00b] and called the *Dixon multiplier matrix* in this paper).

In the case of a generic unmixed polynomial system, any α in the support hull of $\mathcal{A}$ can be used to construct the smallest Dixon multiplier matrix. For convenience, the least degree monomial $x^{\alpha_x}y^{\alpha_y}$ in $\mathcal{A}$ is picked. In section 9, where mixed polynomial systems are discussed, it is shown that the choice of α becomes crucial for generating **good** Dixon multiplier matrices leading to resultants or projection operators with extraneous factors of low degree.

7.1 Exact multiplier matrices

A multiplier matrix of a polynomial system with support $\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle$ constructed using the multiplier sets with supports $\{\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2\}$, has size

$$|\mathcal{X}_0| + |\mathcal{X}_1| + |\mathcal{X}_2| \quad \times \quad |(\mathcal{X}_0 + \mathcal{A}_0) \cup (\mathcal{X}_0 + \mathcal{A}_1) \cup (\mathcal{X}_0 + \mathcal{A}_2)|\,.$$

Assuming that a given multiplier matrix is a resultant matrix, i.e., the determinant of a maximal minor of the matrix is a projection operator, then the matrix is exact if its size (minimum of the number of rows or the number of columns) equals the degree of the resultant.

For the unmixed case, the multiplier matrix is square if $3|\mathcal{X}_0| = |\mathcal{X}_0 + \mathcal{A}_0|$, and is exact if $|\mathcal{X}_0| = 2\,\mathsf{Vol}(\mathcal{A}_0)$. This observation was used in [ZG00] to identify cases when multiplier matrix can be square and exact. Hence, if $|\Delta_{\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \rangle}|$ equals $2\mathsf{Vol}(\mathcal{A}_0)$, then the Dixon multiplier matrix is exact. We have the following consequence:

Theorem 7.1 *For a given unmixed polynomial system $\mathcal{F}$, if its Dixon matrix is exact (in the sense that the resultant of $\mathcal{F}$ is the determinant of the Dixon matrix), then the associated Dixon multiplier matrix is exact as well.*

For the bivariate case, the determinant of the Dixon matrix is the same, irrespective of the variable ordering used in constructing the Dixon polynomial. It is, however, possible to construct two different Dixon multiplier matrices based on different variable orderings. The two multiplier sets are:

$$\mathcal{X}_1 = \Delta_{\mathcal{B}}^{\langle x,y \rangle} - T \quad \text{or} \quad \mathcal{X}_2 = \Delta_{\mathcal{B}}^{\langle y,x \rangle} - T',$$

where $\Delta_{\mathcal{B}}^{\langle x,y \rangle} = \Delta_{\mathcal{B}}$ as discussed in the previous section, and $\Delta_{\mathcal{B}}^{\langle y,x \rangle}$ and T' are the sets constructed in the same way as $\Delta_{\mathcal{B}}$ and T respectively except that the roles of x and y are reversed.

For the unmixed generic bivariate case, if the size of the multiplier set $|\mathcal{X}| = \mu(\mathcal{A}, \mathcal{A}) = 2\mathsf{Vol}(\mathcal{A})$, then the Dixon matrix is exact, implying that its determinant is the resultant. In that case, the Dixon multiplier matrix is also exact.

From the above theorem and Theorem 5.1 in Section 5, we have another key result of the paper:

Theorem 7.2 *Given a generic unmixed bivariate polynomial system $\mathcal{F}$ with support $\mathcal{A}$ and support complement $S = S^{00} \cup S^{01} \cup S^{10} \cup S^{11}$ as well as a point α, the Dixon multiplier matrix M_α constructed using α is exact if and only if each S^k is rectangle and $\alpha \trianglelefteq \mathcal{A}$.*

Since support interior points do not play any role in determining the support of the Dixon polynomial, we get the following corollary of the above theorem.

Corollary 7.2.1 *For a generic unmixed bivariate polynomial system with support $\mathcal{A}$, such that for every $\beta \in \mathbb{N}^2$ if $\beta \trianglelefteq \mathcal{A}$ then $\beta \in \mathcal{A}$, that is, the support $\mathcal{A}$ contains all support hull interior points; let $S = S^{00} \cup S^{01} \cup S^{10} \cup S^{10}$ be the support complement of $\mathcal{A}$. The Dixon multiplier matrix is exact if and only if each S^k is rectangular.*

7.2 Zhang and Goldman's Corner Cut Supports

In [ZG00, Zha00], Zhang and Goldman proposed a method to construct Sylvester-type matrices for the bivariate case. Below, we show how their results follow from our general result above. As will be shown below, our result is stronger since it gives a necessary and sufficient condition on bivariate supports.

Zhang and Goldman [ZG00] defined a *corner-cut* support as a support obtained from a bi-degree support after removing rectangular corners.

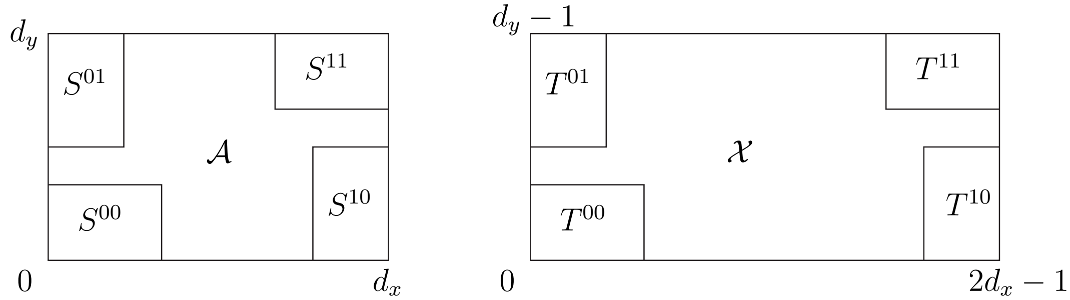

Figure 7: Corner cut support $\mathcal{A}$ and multiplier set $\mathcal{X}$ as in [Zha00]

Definition 7.1 ([ZG00]) *A support $\mathcal{A}$ is called* ***corner-cut*** *if $\mathcal{A} = \mathcal{B} - S$ and all S^k's are rectangles.*

Note that above definition requires that not only all S^k's are rectangles, but also $\mathcal{A}$ contains all of the support interior points.

For an unmixed bivariate polynomial system with a corner-cut support $\mathcal{A}$, Zhang and Goldman proposed to use the following multipliers to construct the resultant matrix:

$$\mathcal{X} = \Delta_{\mathcal{B}}^{\langle y,x\rangle} - T'.$$

In Figure 7, the support $\mathcal{A}$ and the multiplier set $\mathcal{X}$ used by Zhang and Goldman are shown. The Minkowski sum (whose points correspond to the columns of the resultant matrix) is shown in Figure 8. In particular,

$$\begin{aligned} |\mathcal{X}| &= |\Delta_{\mathcal{B}}^{\langle y,x\rangle}| - \left|S^{00}\right| - \left|S^{01}\right| - \left|S^{10}\right| - \left|S^{11}\right| = 2\,\mathsf{Vol}(\mathcal{A}) \quad \text{and} \\ |\mathcal{A} + \mathcal{X}| &= 3|\Delta_{\mathcal{B}}^{\langle y,x\rangle}| - 3\left|S^{00}\right| - 3\left|S^{01}\right| - 3\left|S^{10}\right| - 3\left|S^{11}\right| = 3\left|\mathcal{X}\right|. \end{aligned}$$

The matrix defined by Zhang and Goldman's construction is square, and its size is exact in the sense that each polynomial appears in the matrix as many times as the number of toric roots of the other two polynomials. It was shown in [ZG00] that these matrices are nonsingular in the generic case. Hence, their determinant is the resultant.

A corner-cut support $\mathcal{A}$ satisfies the condition in Theorem 7.2, giving

Corollary 7.2.2 *Given a generic unmixed polynomial system $\mathcal{F}$ with a corner-cut support $\mathcal{A}$, the determinant of the Dixon multiplier matrix constructed using multipliers from $\mathcal{X}$ is its resultant.*

It is also possible to use the multipliers

$$\mathcal{X}' = \Delta_{\mathcal{B}}^{\langle x,y\rangle} - T,$$

giving the exact resultant.

The condition in Corollary 7.2.1 is weaker in contrast than the one required by Zhang and Goldman. Even if the support $\mathcal{A}$ of a generic unmixed bivariate polynomial system is not corner-cut, but the support $\mathcal{A}'$ including all support interior points of $\mathcal{A}$ is corner-cut, even then the resultant can be computed exactly using the Dixon multiplier matrix construction. Furthermore, this is a necessary and sufficient condition for the determinant of the associated Dixon multiplier matrix to be the resultant.

Another immediate corollary of this result is that if $\mathcal{A}$ is is not corner-cut, the determinant of the Dixon multiplier matrix constructed using multipliers from $\mathcal{X}$ (or $\mathcal{X}'$) is a nontrivial multiplier of its resultant (in other words, there is an extraneous factor).

The notion of a corner-cut support cannot be naturally extended to polynomial systems with more than two variables, as corner cut construction does not yield exact matrices for some simple 3 dimensional supports. For example, an unmixed polynomial system with support $\{(0,0,0),(1,0,0),(0,1,0),(0,0,2),(1,0,2),(0,1,2)\}$ can be thought of as corner-cut, as the rectangular corner is missing at points $(1,1,0),(1,1,1)$ and $(1,1,2)$, yet

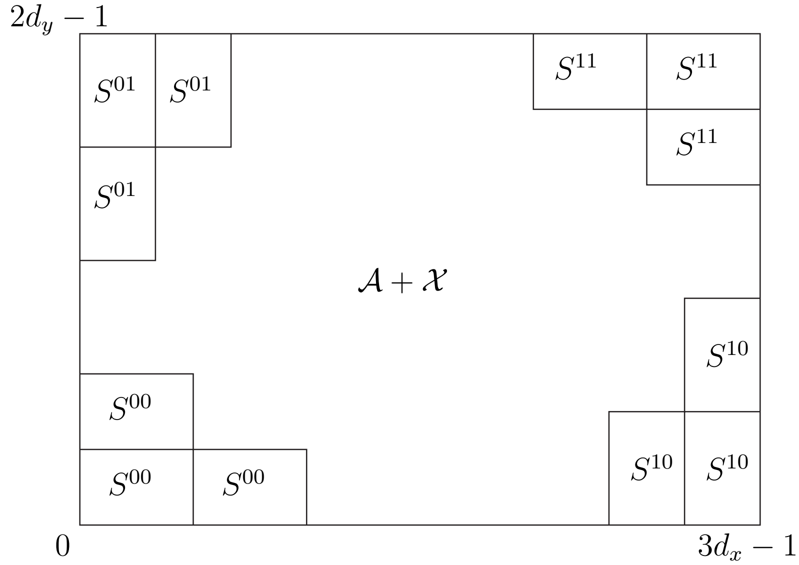

Figure 8: The Minkowski Sum $\mathcal{A} + \mathcal{X}$ as in [Zha00]

the Dixon multiplier matrix will result in extraneous factors, if care is not taken to choose appropriate variable order while constructing Dixon polynomial. This raises an interesting open question: given a corner-cut support in 3 dimensions (generalized in the natural way), does there always exist a variable order making the Dixon multiplier matrix exact?

8 Examples: Unmixed Case

In this section, we compare a number of different methods on generic unmixed bivariate polynomial systems. First five examples in Table 1 are from [DE01a], and the sixth example has its support as given in Figure 2. Since the method proposed by [ZG00] (where natural generalization is taken for non-corner-cut supports) and the proposed matrix M_α are the same for *full* unmixed supports, only one column is shown for both in the table.

Column labelled by $\deg R$ denotes the resultant total degree, which is $\deg_{f_0} R + \deg_{f_1} R + \deg_{f_2} R$ where $\deg_{f_i} R$ is the degree of coefficients of monomials in f_i in the resultant R. Other columns are identified by references to articles in which the respective method was proposed. The entries in these columns show the extraneous factor degree, that is the degree of projection operator minus the resultant degree. A detailed explanation of examples follows.

Table 1: Comparison of resultant matrices on bivariate unmixed systems

Ex.	deg R	[CE00][a]		[DE01a][a]		[ZG00][b] and M_α	
		Matrix	Extra	Matrix	Extra	Matrix	Extra
1	$3\,n^2$	$\frac{3}{2}n(3n-1)$	$\frac{3}{2}n(n-1)$	$\frac{9}{2}n(n-1)+1$	$\frac{3}{2}n(n-3)+3$	$4n^2-n$	n^2-n
2	$6\,n_1n_2$	$9n_1n_2$	$3n_1n_2$	$(3n_1-1)(3n_2-1)$	$3(n_1-1)(n_2-1)$	$6\,n^2$	0
3	12	15	3	10	0	12	0
4	18	25	7	22	6	18	0
5	57	75	18	76	21	59	2
6	111	149	38	141	32	117	6

[a] Random algorithm, minimal of 10 runs chosen (Exs. 1 & 2 are reported in [DE01a])
[b] Generalized to non-corner cut supports

As can be seen from the table, the Dixon multiplier matrices often compute resultant exactly and in the cases where they do not give the exact resultants, they yield projection operators of the smallest degrees (with the exception of full homogeneous systems). It can be shown that the worst case happens for full homogeneous systems, where the cut-off corner is the least similar to a rectangle.

1. **Homogeneous** (unmixed) polynomial system of degree n:

$$f_0(x,y) = \sum_{i+j\le n} a_{ij}x^iy^j, \qquad f_1(x,y) = \sum_{i+j\le n} b_{ij}x^iy^j, \qquad f_2(x,y) = \sum_{i+j\le n} c_{ij}x^iy^j.$$

   The mixed volume of any two polynomials is n^2, the Bezout bound. The degree of the resultant is $3n^2$.

   This system can be computed using Macaulay resultant formulation exactly, where extraneous factor is readily identified in the determinant of Macaulay matrix.

2. **Bi-homogeneous** (unmixed, corner cut) polynomial system of degree n_1, n_2:

$$f_0(x,y) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_{ij}x^iy^j, \qquad f_1(x,y) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} b_{ij}x^iy^j, \qquad f_2(x,y) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} c_{ij}x^iy^j.$$

   The mixed volume of any two polynomials is $2n_1n_2$. The degree of the resultant is $6n_1n_2$.

3. **Examples from [CDS98]** (unmixed, corner cut):

$$\begin{aligned} f_0(x,y) &= a_{00} + a_{01}y + a_{10}x + a_{11}xy + a_{12}xy^2 + a_{13}xy^3, \\ f_1(x,y) &= b_{00} + b_{01}y + b_{10}x + b_{11}xy + b_{12}xy^2 + b_{13}xy^3, \\ f_2(x,y) &= c_{00} + c_{01}y + c_{10}x + c_{11}xy + c_{12}xy^2 + c_{13}xy^3. \end{aligned}$$

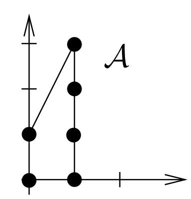

Figure 9: Support of example 3

   This problem is given as an example in [CDS98] of the Chow form of a Hilzebruch surface. It is an unmixed problem, where any two polynomials have the mixed volume of 4. Notice that this problem has a corner-cut support.

4. **Example from [DE01b]** (unmixed, corner cut):

$$\begin{aligned} f_0 &= a_{00} + a_{10}x + a_{01}y + a_{12}xy^2 + a_{21}x^2y + a_{22}x^2y^2, \\ f_1 &= b_{00} + b_{10}x + b_{01}y + b_{12}xy^2 + b_{21}x^2y + b_{22}x^2y^2, \\ f_2 &= c_{00} + c_{10}x + c_{01}y + c_{12}xy^2 + c_{21}x^2y + c_{22}x^2y^2. \end{aligned}$$

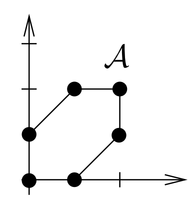

Figure 10: Support of example 4

   The mixed volume of any two polynomials is 6; therefore, the degree of the resultant is 18. Moreover, the problem is unmixed and corner-cut; therefore, the Dixon method, the Dixon multiplier matrix method and [ZG00] have exact matrices for this problem.

   It is included in [DE01b]; it is interesting because the hybrid method proposed in [DE01b] does not produce an exact resultant matrix.

5. **Example from [DE01a]** (unmixed)

$$
\begin{aligned}
f_0(x,y) &= a_{10}x + a_{21}x^2y + a_{03}y^3 + a_{15}xy^5 + a_{25}x^2y^5 + a_{33}x^3y^3 + a_{34}x^3y^4,\\
f_1(x,y) &= b_{10}x + b_{21}x^2y + b_{03}y^3 + b_{15}xy^5 + b_{25}x^2y^5 + b_{33}x^3y^3 + b_{34}x^3y^4,\\
f_2(x,y) &= c_{10}x + c_{21}x^2y + c_{03}y^3 + c_{15}xy^5 + c_{25}x^2y^5 + c_{33}x^3y^3 + c_{34}x^3y^4.
\end{aligned}
$$

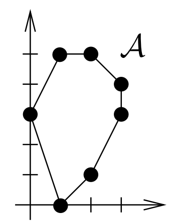

Figure 11: Support of example 5

This example appeared in [DE01a] as a demonstration for the hybrid method proposed in that paper for bivariate systems. The mixed volume of any two polynomials is 19; hence, the resultant degree is 57.

6. **Example from Figure 2**

$$
\begin{aligned}
f_0 &= a_{02}y^2 + a_{12}xy^2 + a_{21}x^2y + a_{22}x^2y^2 + a_{30}x^3 + a_{30}x^3 + a_{31}x^3y + a_{32}x^3y^2\\
&\quad + a_{54}x^5y^4 + a_{55}x^5y^5 + a_{56}x^5y^6 + a_{65}x^6y^5 + a_{66}x^6y^6 + a_{75}x^7y^5,\\
f_1 &= b_{02}y^2 + b_{12}xy^2 + b_{21}x^2y + b_{22}x^2y^2 + b_{30}x^3 + b_{30}x^3 + b_{31}x^3y + b_{32}x^3y^2\\
&\quad + b_{54}x^5y^4 + b_{55}x^5y^5 + b_{56}x^5y^6 + b_{65}x^6y^5 + b_{66}x^6y^6 + b_{75}x^7y^5,\\
f_2 &= c_{02}y^2 + c_{12}xy^2 + c_{21}x^2y + c_{22}x^2y^2 + c_{30}x^3 + c_{30}x^3 + c_{31}x^3y + c_{32}x^3y^2\\
&\quad + c_{54}x^5y^4 + c_{55}x^5y^5 + c_{56}x^5y^6 + c_{65}x^6y^5 + c_{66}x^6y^6 + c_{75}x^7y^5.
\end{aligned}
$$

This system has 2-fold mixed volume of $\langle 37, 37, 37\rangle = 111$; its resultant degree is thus 111. It is not corner-cut as the support complements are not rectangular. This example demonstrates the fact that with a small increase in the size of the support, the resultant grows quite fast.

Based on the above table, it can be said that the Dixon multiplier matrices generate projection operators with low degree extraneous factors in contrast to other methods. Further, the Dixon multiplier matrices turn out to be even more effective for computing projection operators from mixed polynomial systems as discussed in the next section, whereas most other methods are not easily generalizable to arbitrary mixed bivariate systems.

9 Mixed Polynomial Systems

We show below that the Dixon multiplier matrix construction is especially effective for mixed polynomial systems. This construction depends upon on generating a multiplier set for each polynomial, and is determined by the presence (or absence) of monomials in the support of the polynomials in the polynomial system. As should be evident from the above discussion, the multiplier sets determine the size of the Dixon multiplier matrix and hence, the degree of a projection operator.

Consider a mixed generic polynomial bivariate system $\mathcal{F} = \{f_0, f_1, f_2\}$:

$$
f_0 = \sum_{\alpha\in\mathcal{A}_0} a_{i,j}\; x^{\alpha_x}y^{\alpha_y}, \qquad f_1 = \sum_{\beta\in\mathcal{A}_1} b_{i,j}\; x^{\beta_x}y^{\beta_y}, \qquad f_2 = \sum_{\gamma\in\mathcal{A}_2} c_{i,j}\; x^{\gamma_x}y^{\gamma_y},
$$

where $\mathcal{A}_i$ is a support of f_i. Since we are only interested in toric roots, we can pre-multiply each polynomial by any monomial, that is, instead, it is possible to consider

$$
f_0 = x^{t_{0,x}}y^{t_{0,y}} \sum_{\alpha\in\mathcal{A}_0} a_{i,j}\; x^{\alpha_x}y^{\alpha_y}, \qquad f_1 = x^{t_{1,x}}y^{t_{1,y}} \sum_{\beta\in\mathcal{A}_1} b_{i,j}\; x^{\beta_x}y^{\beta_y}, \qquad f_2 = x^{t_{2,x}}y^{t_{2,y}} \sum_{\gamma\in\mathcal{A}_2} c_{i,j}\; x^{\gamma_x}y^{\gamma_y},
$$

for $t = \langle t_0, t_1, t_2\rangle$ where $t_i \in \mathbb{N}^2$, or equivalently,

$$
f_0 = \sum_{\alpha\in\mathcal{A}_0+t_0} a_{i,j}\; x^{\alpha_x}y^{\alpha_y}, \qquad f_1 = \sum_{\beta\in\mathcal{A}_1+t_1} b_{i,j}\; x^{\beta_x}y^{\beta_y}, \qquad f_2 = \sum_{\gamma\in\mathcal{A}_2+t_2} c_{i,j}\; x^{\gamma_x}y^{\gamma_y}.
$$

For mixed polynomial systems, the construction of a **good** Dixon multiplier matrix (in the sense that its determinant gives the projection operator of the least degree) is sensitive to the choice of m as well as the translation vector t for the supports. Choosing an appropriate m and t can be formulated as an optimization problem in which the size of the support of each $\theta_i(m)$ and hence, the multiplier set for each f_i, is minimized.

For any given support point $\alpha = (\alpha_x, \alpha_y)$, let $m = x^{\alpha_x} y^{\alpha_y}$; then $\Delta_{\langle \{\alpha\}, \mathcal{A}_1 + t_1, \mathcal{A}_2 + t_2 \rangle}$ is the support of $\theta_0(m)$; similarly, $\Delta_{\langle \mathcal{A}_0 + t_0, \{\alpha\}, \mathcal{A}_2 + t_2 \rangle}$ is the support of $\theta_1(m)$ and $\Delta_{\langle \mathcal{A}_0 + t_0, \mathcal{A}_1 + t_1, \{\alpha\} \rangle}$ is the support of $\theta_2(m)$. Let

$$\Phi_0(\alpha, t) = |\Delta_{\langle \{\alpha\}, \mathcal{A}_1 + t_1, \mathcal{A}_2 + t_2 \rangle}|, \qquad \Phi_1(\alpha, t) = |\Delta_{\langle \mathcal{A}_0 + t_0, \{\alpha\}, \mathcal{A}_2 + t_2 \rangle}| \qquad \text{and} \qquad \Phi_2(\alpha, t) = |\Delta_{\langle \mathcal{A}_0 + t_0, \mathcal{A}_1 + t_1, \{\alpha\} \rangle}|.$$

Since $\Phi_i(\alpha, t)$ represents the number of rows corresponding to polynomial $x^{t_{i,x}} y^{t_{u,y}} f_i$ in the Dixon multiplier matrix, the goal is to find α and $t = \langle t_0, t_1, t_2 \rangle$ such that

$$\Phi(\alpha, t) = \Phi_0(\alpha, t) + \Phi_1(\alpha, t) + \Phi_2(\alpha, t)$$

is minimized; that is, the size of the entire Dixon multiplier matrix is minimized so as to minimize the degree of the extraneous factor. One can choose to minimize particular $\Phi_i(\alpha, t)$ in the hope of having coefficients of f_i appearing with the smallest degree in the projection operator.

Example: Consider the following polynomial system:

$$\begin{aligned} f_0 &= a_{00} + a_{10}x + a_{01}y, \\ f_1 &= b_{02}y^2 + b_{20}x^2 + b_{31}x^3y, \\ f_2 &= c_{00} + c_{12}xy^2 + c_{21}x^2y. \end{aligned}$$

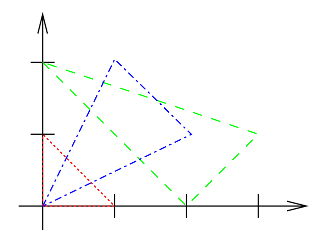

Figure 12: Mixed example.

This generic polynomial system has 2-fold mixed volume of $\langle 8, 3, 4 \rangle = 15$; hence, the optimal multiplier matrix is 15×15, containing 8 rows from polynomial f_0, 3 rows from f_1 and 4 rows from f_2. Figure 12 shows the overlaid supports of these polynomials.

To construct the Dixon multiplier matrix, if we choose $\alpha = (0, 0)$ and $t = \langle (0, 0), (0, 0), (0, 0) \rangle$ as in [CK00b],

$$\Phi_0(\alpha, t) = 9, \quad \Phi_1(\alpha, t) = 4, \quad \Phi_2(\alpha, t) = 5,$$

and the Dixon multiplier matrix has $\Phi(\alpha, t) = 18$ rows. In fact, if $t = \langle (0, 0), (0, 0), (0, 0) \rangle$ then the best choice for α is from $\{(0, 0), (0, 1), (1, 0)\}$, each one producing a 18×18 Dixon multiplier matrix. In other words, an extraneous factor of at least degree 3 is generated using the Dixon multiplier matrix no matter what multiplier monomial is used if supports are not translated.

On the other hand, if $t = \langle (2, 1), (0, 0), (1, 0) \rangle$ and $\alpha \in \{(2, 1), (2, 2), (3, 1)\}$,

$$\Phi_0(\alpha, t) = 8, \quad \Phi_1(\alpha, t) = 3, \quad \Phi_2(\alpha, t) = 4,$$

and $\Phi(\alpha, t) = 15$, i.e., the Dixon multiplier matrix is optimal. Figure 13 shows the translated supports. This example also illustrates that it is possible to get exact resultant matrices if supports are translated even when untranslated supports have a nonempty intersection.

The Dixon matrix for the original polynomial system is of size 9×9, whereas for the translated polynomial system, the Dixon matrix is of size 8×8. In both cases, there are extraneous factors of degree 12 and 9, respectively. In fact, it can be shown that in generic mixed cases, the size of the Dixon matrix is $\max(\Phi_0, \Phi_1, \Phi_2)$ when monomial in the construction is appropriately chosen (see [CK02b]).

As illustrated from the above example, the Dixon multiplier matrix as well as the Dixon matrix are sensitive to a translation vector t. Since the mixed volume is invariant under translation t, most resultant methods in which matrices are constructed using supports are also invariant to different values of t. Moreover, since the Dixon multiplier matrix is sensitive to the choice of α,(whereas the Dixon matrix is not), it is possible to further optimize the size of the Dixon multiplier matrix by properly selecting the multiplier monomial.

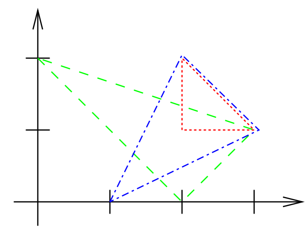

Figure 13: Translated example.

9.1 Searching the Appropriate Monomial for Constructing Multiplier Matrix

For the bivariate case, the evaluation of $\Phi_i(\alpha, t)$ is not too costly; finding the optimal α and t can be done by an exhaustive search procedure. The following observations will be used however to limit the search somewhat.

1. Let $\breve{\mathcal{A}}^{-i} = \texttt{SupportHull}(t_j + \mathcal{A}_j \cup t_k + \mathcal{A}_k)$, where $j, k \neq i$ and $i, j, k \in \{0, 1, 2\}$; by the results from [CK02b],
$$\Delta_{\langle\{\alpha\}, t_j+\mathcal{A}_j, t_k+\mathcal{A}_k\rangle} \subseteq \Delta_{\breve{\mathcal{A}}^{-i}} \qquad \text{when} \quad \alpha \in \breve{\mathcal{A}}^{-i}.$$
Let $\mu = \langle \mu_0, \mu_1, \mu_2 \rangle$ be the 2-fold mixed volume of the supports. Note that for an optimal matrix, $\Phi_i(\alpha, t) = \mu_i$, for $i = 0, 1, 2$. Hence in general we have
$$\mu_i \quad \leq \quad \Phi_i(\alpha, t) \quad \leq \quad |\Delta_{\breve{\mathcal{A}}^{-i}}|.$$
Since it is difficult to minimize $\Phi_i(\alpha, t)$ without exhaustive search, we will try to minimize the upper bound $|\Delta_{\breve{\mathcal{A}}^{-i}}|$, as in that case once a translation vector t is fixed, then the choice for α is clear.

2. To choose α, once t is fixed, compute $\breve{\mathcal{A}}^{-0}$, $\breve{\mathcal{A}}^{-1}$ and $\breve{\mathcal{A}}^{-2}$ and choose $\alpha \in \breve{\mathcal{A}}^{-0} \cap \breve{\mathcal{A}}^{-1} \cap \breve{\mathcal{A}}^{-2}$.

3. The translation vector t should be so chosen that $\breve{\mathcal{A}}^{-0} \cap \breve{\mathcal{A}}^{-1} \cap \breve{\mathcal{A}}^{-2} \neq \oslash$ and sizes of $|\Delta_{\breve{\mathcal{A}}^{-i}}|$ for unmixed $\breve{\mathcal{A}}^{-i}$, for $i \in \{0, 1, 2\}$, are minimal as they are upper bounds on $\Phi_i(\alpha, t)$.

For $t = \langle t_0, t_1, t_2 \rangle$, only two of the three t_i's need to be found, as one support can be arbitrarily placed and other two need to be optimized. If t_0 is fixed, t_1 can be computed so that $|\Delta_{\breve{\mathcal{A}}^{-2}}|$ is minimal; after that, t_2 can be computed so as to minimize $|\Delta_{\breve{\mathcal{A}}^{-1}}|$. Further, $|\Delta_{\breve{\mathcal{A}}^{-0}}|$ will be determined once t_1, t_2 are chosen. After t has been determined, then α can be selected.

As an example, consider the following polynomial system.

$$\begin{aligned} f_0 &= a_{00} + a_{20}x^2 + a_{3,6}x^3y^6 + a_{7,6}x^7y^6, \\ f_1 &= b_{10}x + b_{07}y^7 + b_{2,9}x^2y^9 + b_{3,9}x^2y^9, \\ f_2 &= c_{00} + c_{25}x^2y^5 + c_{8,4}x^8y^4. \end{aligned}$$

Its support is:

$$\begin{aligned} \mathcal{A}_0 &= \{(0,0), (2,0), (3,6), (7,6)\}, \\ \mathcal{A}_1 &= \{(1,0), (0,7), (2,9), (3,9)\}, \quad \text{and} \\ \mathcal{A}_2 &= \{(0,0), (2,5), (8,4)\}. \end{aligned}$$

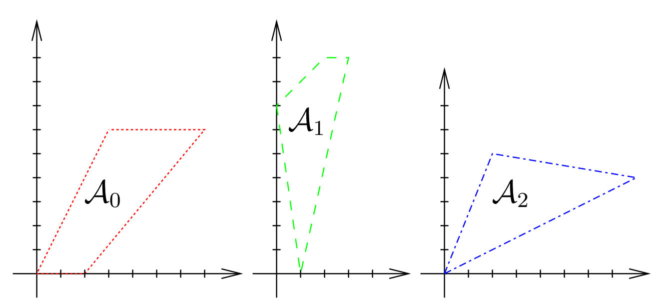

Figure 14: Support of example 5

The supports of the above polynomial system has the mixed volume of $\langle 75, 51, 63 \rangle = 189$; hence, the optimal matrix is of size 189. t_0 is fixed at $(3, 3)$, thus giving enough space to choose t_1 and t_2, without getting into negative coordinates.

From Table 2, the best choice for t_1 is $(4, 0)$, for which the upper bound on $\Phi_2(\alpha, t)$ is 65. From Table 3, the best choice for t_2 is $(3, 4)$ or $(4, 4)$ with the upper bound on $\Phi_1(\alpha, t)$ being 55. Fixing $t_1 = (4, 0)$ and determining the size of $\Delta_{\breve{\mathcal{A}}^{-0}}$, the upper bound on Φ_0 is 77 and 75, respectively, for two different values $t_2 = (3,4)$ and $t_2 = (4,4)$. Hence, with a choice of $t = \langle (3,3), (4,0), (4,4) \rangle$,

$$75 \le \Phi_0(\alpha, t) \le 75, \qquad 51 \le \Phi_1(\alpha, t) \le 55, \qquad 63 \le \Phi_2(\alpha, t) \le 65.$$

No	t_1	$\lvert\Delta_{\breve{\mathcal{A}}^{-2}}\rvert$	No	t_1	$\lvert\Delta_{\breve{\mathcal{A}}^{-2}}\rvert$
1.	(2,0)	82	16.	(5,2)	71
2.	(3,0)	71	17.	(6,2)	71
3.	(4,0)	65	18.	(7,2)	77
4.	(5,0)	67	19.	(2,3)	88
5.	(6,0)	69	20.	(3,3)	80
6.	(7,0)	75	21.	(4,3)	74
7.	(2,1)	84	22.	(5,3)	74
8.	(3,1)	74	23.	(6,3)	74
9.	(4,1)	68	24.	(7,3)	80
10.	(5,1)	69	25.	(2,4)	98
11.	(6,1)	70	26.	(3,4)	90
12.	(7,1)	76	27.	(4,4)	84
13.	(2,2)	86	28.	(5,4)	83
14.	(3,2)	77	29.	(6,4)	82
15.	(4,2)	71	30.	(7,4)	84

2: Choosing translation vector t_1

No	t_2	$\lvert\Delta_{\breve{\mathcal{A}}^{-1}}\rvert$	No	t_2	$\lvert\Delta_{\breve{\mathcal{A}}^{-1}}\rvert$
1.	(0,0)	101	16.	(0,3)	80
2.	(1,0)	95	17.	(1,3)	71
3.	(2,0)	92	18.	(2,3)	62
4.	(3,0)	93	19.	(3,3)	57
5.	(4,0)	100	20.	(4,3)	58
6.	(0,1)	94	21.	(0,4)	80
7.	(1,1)	87	22.	(1,4)	70
8.	(2,1)	82	23.	(2,4)	60
9.	(3,1)	81	24.	(3,4)	55
10.	(4,1)	86	25.	(4,4)	55
11.	(0,2)	87	26.	(0,5)	86
12.	(1,2)	79	27.	(1,5)	75
13.	(2,2)	72	28.	(2,5)	64
14.	(3,2)	69	29.	(3,5)	60
15.	(4,2)	72	30.	(4,5)	61

3: Choosing translation vector t_2

In particular, with this choice of t, we are guaranteed to have a projection operator with an extraneous factor of at most of degree 6, and the degree of projection operator in terms of the coefficients of the first polynomial is exact.

The multiplier monomial α is chosen from $\breve{\mathcal{A}}^{-0} \cap \breve{\mathcal{A}}^{-1} \cap \breve{\mathcal{A}}^{-2} = \{(4,4), (5,4), (5,3), (6,9), (7,9)\}$, where

$$\Phi_0(\alpha, t) = 75, \qquad \Phi_1(\alpha, t) = 53, \qquad \Phi_2(\alpha, t) = 65.$$

Hence, the Dixon multiplier matrix has 193 rows, and the degree of the extraneous factor in its projection operator is at most 4. In case, the matrix is singular, the degree of the extraneous factor can be smaller, depending upon the maximal minor selected from the resultant matrix.

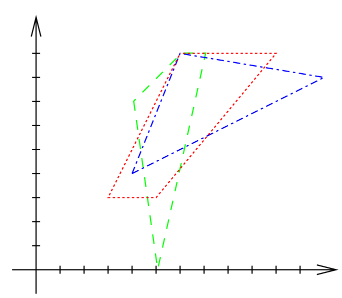

Figure 15: Optimal t

10 Examples

In this section, we discuss a family of bivariate systems discussed in the literature [DE01a], and the performance of different algorithms in generating resultant matrices. The details about the examples are given after the table. As in the table on examples in Section 8, the column *deg R* gives the degree of the resultant. Each method is identified by the paper in which it appeared. The last two columns are for the two methods based on the Dixon formulation discussed in this paper. The column labelled $|\Delta_{\mathcal{A}}|$ gives the size of the Dixon matrix. The reader should recall that the entries in the Dixon matrix are 3×3 determinants expressed in the coefficients of the terms in the polynomials. For other methods, matrix entries are mostly zeros or coefficients of terms in the polynomials. The last column in the table is the size of the Dixon multiplier matrix.

The degree of the projection operators cannot be determined from the matrix sizes in the case of [DE01a] and the Dixon matrix (the Θ column) as some of the matrix entries are different from coefficients of terms in polynomials. For the Dixon matrix, the degree of projection operator is $3|\Theta|$; for the method in [DE01a], the degree of the projection operator is 2 more than the matrix size.

From Table 4, it is clear that the Dixon multiplier matrix method produces smaller extraneous factors; in almost all examples, it computes projection operators of the lowest degrees, often giving exact resultants. The method also turns out to be computationally less expensive for extracting a projection operator.

Table 4: Comparison of resultant matrices

Ex.	$\deg R$	[CE00][a]		[ZG00][b]		[DE01a][a]		Θ		M_α	
		Size	Extra	Size	Extra	Size	Extra	Size	Extra	Size	Extra
1	5	5	0	6	1	4	1	2	1	5	0
2	7	12	5	15	8	7	2	4	5	7	0
3	24	35	11	30	6	22	0	8	0	24	0
4	15	15	0	24	9	28	15	8	9	15	0
5	189	194	5	264	75	381	194	88	75	192[c]	3

[a] Random algorithm; minimal of 10 runs chosen (Exs. 1, 2 & 3 are reported in [DE01a])
[b] Generalized to non-corner cut supports, where the union of the three supports is used as the support of the system.
[c] The Dixon multiplier matrix in this case is of size 193×192.

We should note that [DE01a] and [ZG00] are designed for unmixed polynomials systems, and hence are not well suited for mixed systems. They have been adapted to mixed as in [DE01a] and included for comparison purposes only. With the new results from authors of [DE01a], it is possible to identify extraneous factor in projection operator as a determinant of certain minor of resultant matrix. It is not clear if such minor exists for the Dixon multiplier matrices.

Constructing smaller resultant matrices is not only an attempt to tackle with the problem of extraneous factors, but this also improves complexity. Since resultant matrices are symbolic, computing determinants of such matrices is often of exponential complexity in the matrix size. So any heuristic or optimization leading to matrices of smaller size are to be preferred.

1. **Example from [ZG00]** (mixed) This example appeared earlier in section 9. It served as an example in [DE01a] and [ZG00].

$$f_0(s,t) = 2s + t, \qquad f_1(s,t) = st + st^2, \qquad f_2 = s^2t + 2t.$$

   The mixed volume is $\langle 2,2,1\rangle$ (i.e., $\mu(\mathcal{A}_1,\mathcal{A}_2) = 2$, $\mu(\mathcal{A}_0,\mathcal{A}_2) = 2$ and $\mu(\mathcal{A}_0,\mathcal{A}_1) = 1$). The degree of the resultant is 5.

2. **Example from [Man92]** (Mixed):

$$\begin{aligned} f_0(x,y) &= a_{10}x + a_{20}x^2 + a_{01}y, \\ f_1(x,y) &= b_{10}x + b_{02}y^2 + b_{01}y, \\ f_2(x,y) &= c_{10}x + c_{11}xy + c_{01}y. \end{aligned}$$

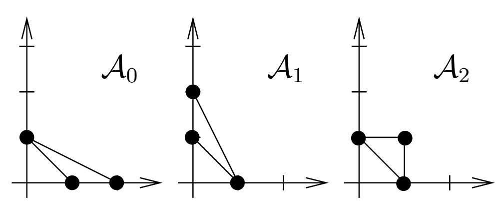

Figure 16: Support of example 2

   This problem is about surface parameterization [Man92]. Its BKK bound is $\langle 2,2,3\rangle = 7$.

3. **Example from [GS01]** (mixed):

$$\begin{aligned} f_0 &= a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{21}x^2y + a_{22}x^2y^2 + a_{31}x^3y + a_{32}x^3y^2, \\ f_1 &= b_{00} + b_{10}x + b_{01}y + b_{11}xy + b_{21}x^2y + b_{22}x^2y^2 + b_{31}x^3y, \\ f_2 &= c_{00} + c_{10}x + c_{01}y + c_{11}xy + c_{21}x^2y + c_{31}x^3y + c_{32}x^3y^2. \end{aligned}$$

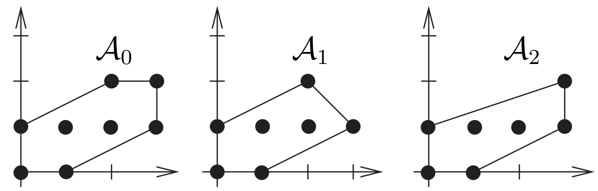

Figure 17: Support of example 3

This polynomial system is defined in [GS01] to study the self intersections of a parameterized surface. Interestingly, this problem has the BKK bound of $\langle 8, 8, 8\rangle = 24$, which is the same as though this system was unmixed whose support equals the support of the first polynomial which is also the union of the supports of the other two polynomials.

4. **Example from section 9**.

$$\begin{aligned} f_0 &= a_{00} + a_{10}x + a_{01}y, \\ f_1 &= b_{02}y^2 + b_{20}x^2 + b_{31}x^3y, \\ f_2 &= c_{00} + c_{12}xy^2 + c_{21}x^2y. \end{aligned}$$

This polynomial system has 2-fold mixed volume of $\langle 8, 3, 4\rangle = 15$; hence the degree of the resultant is 15.

5. **Example from section 9.1**.

11 Generalization to Multivariate Polynomial Systems

We have identified necessary and sufficient conditions on the support of an unmixed bivariate polynomial system such that the methods based on the Dixon resultant formulation can compute its resultant exactly. When this cannot be done, the degree of the projection operator can be predicted, from which the degree of the extraneous factor appearing in it can be computed. Knowing the degree of the extraneous factor in a projection operator is helpful in identifying the resultant in the projection operator. A method for computing the Dixon multiplier matrices based on the Dixon formulation was proposed; unlike the Dixon matrices, the Dixon multiplier matrices are Sylvester-type in the sense that matrix entries are either 0 or coefficients of terms in the polynomial systems. These results are thus strict generalizations of the related results reported in [Chi01, ZG00, Zha00].

For mixed bivariate systems, heuristics were developed for translating supports and selecting a monomial for computing the Dixon multiplier matrices so that projection operators computed from these matrices are either resultants or besides the resultants, they have extraneous factors of low degrees.

The above results still do not lead to precisely identifying the extraneous factor as is known in the case of eliminating a single variable. We plan to investigate this issue next.

For non-generic polynomial systems for which the number of toric roots is still the BKK bound, the degree of extraneous factor in a projection operator cannot be estimated. It appears that the discrepancy between the BKK bound and the size of Dixon matrix is due to the difference in the volume of the Newton polytope and the size of the corresponding support. Experimental evidence suggests that the coefficients of terms in polynomials also play a role in determining the support of the Dixon polynomial; this is also reflected in the formula for the Dixon polynomial based on the Cauchy-Binet formula. We are interested in analyzing whether the genericity requirements for obtaining the BKK bound is sufficient to preclude any role the coefficients of terms in a polynomial system play in determining the support of the Dixon polynomial and hence, the size of Dixon matrix.

The focus of this paper has been on bivariate systems. We are interested in generalizing these results – particularly the concepts of a support-interior point, support hull and corner-cut support in an arbitrary dimension. As illustrated above, it can be shown that the determinant of the Dixon multiplier matrix is not exactly the resultant for a tri-variate generic unmixed system even if its support is corner-cut.[5] However, a

[5] The notion of a corner-cut support is assumed to be generalized to arbitrary dimensions in an obvious way.

necessary and sufficient condition on supports based on exclusion of support-interior points seems plausible, and is under further investigation

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